Basic Concepts for Unfying Queries of Database and Retrieval Systems

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Abstract

Traditional database query languages are based on set theory and crisp logic. Many applications, however, need additionally retrieval-like queries returning the probability of a result object being relevant to a given query object. Traditionally, retrieval systems estimate relevance by exploiting hidden semantics whereas query processing in database systems relies mainly on term matching. Thus, completely different mechanisms were developed for database and information retrieval. As result, there is a lack of support for queries which involve both retrieval and database search terms. In this paper we develop basic concepts for a unifying framework based on quantum mechanics and quantum logic. Rijsbergen already discussed in his book the strong relation between quantum mechanics and information retrieval. The goal of this paper is to discuss the relation of database queries to quantum mechanics. Here, we develop the mapping of `select` operations as well as conjunction, disjunction, and negation as part of the relational calculus to quantum mechanics and quantum logic.

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Chapter 1

Introduction

In some application areas, e.g. in multimedia applications, expressing an information need requires often a mixture of traditional database queries and retrieval-like search terms. Retrieval functionality, for example, is required if database objects are searched by using some notion of similarity. For example, consider the following query: *'Retrieve all van Gogh images that are similar in form and color to the given image'*. This example query involves two conjunctively combined similarity predicates and one database selection on the attribute `painter`. Information retrieval returns result objects equipped with a value from $[0, 1]$ which is usually interpreted as estimated probability of the corresponding object being relevant to the query.

Traditional database query languages like the relational calculus offer Boolean algebra operators to construct complex search conditions from atomic conditions. However, such operators are not able to deal with values from $[0, 1]$ which are returned from a retrieval system in response of a retrieval query.

In database query processing, deciding whether a database object belongs to a query result is based on term matching. Thus, a database system knows exactly when an object fulfills a given query. This is not possible for retrieval systems. Usually, the information required for an exact decision is not explicitly available forcing the retrieval system to estimate the probability of relevance. Historically, for database querying and information retrieval different concepts have been developed. This causes a special problem for complex queries which involve retrieval as well as database query elements. So far, no satisfactory solution was found. One possible solution approach is the usage of fuzzy logic in order to combine truth values from database query processing with results from retrieval queries. However, in such approaches fuzzy conjunction and disjunction operate on the truth values alone and do not take the semantics of the underlying search conditions into consideration.

Alternatively, in this work we investigate whether quantum mechanics and quantum logic can provide us a unifying framework for querying in databases and retrieval systems. Quantum mechanics provides a means to completely describe every closed physical system. It comes with an own mathematical formalism. This formalism is attractive for our problem since it combines in a very elegant way concepts from the field of geometry (linear algebra and Hilbert space), logic (quantum logic as a non-standard logic) and probability theory. In our approach we try to map logic concepts used in database query languages into quantum logic. Retrieval systems estimate probability values and a prominent retrieval model is the vector space model. Those concepts necessary for information retrieval are also available in the theory of quantum mechanics.

The focus of this work is on mapping concepts from database query processing to quantum processing and to establish in this way a connection to information retrieval. The long-term goals of our work are:

- support of queries which involve retrieval and database query search conditions and
- getting a deep understanding of the relationship between both fields in order to enhance the power of database query language and of information systems:
  - *database concepts for information retrieval:* Database query languages provide means to formulate complex queries by using first order logic. We will investigate (not in this work) how those concepts can be utilized to formulate complex retrieval queries.
  - *information retrieval concepts for database query processing:* Information retrieval deals with probability values. We will investigate how to incorporate the notion of impreciseness into database query processing.

Chapter 2

Related Work

The development of quantum mechanics dates back to the begin of the 20th century. This topic was strongly influenced by famous physicists like Planck, Bohr, Schröder, and Heisenberg. It deals with specific phenomena like uncertainty of measurements in closed physical systems. The so-called Gleason-theorem [Gle57] shows the connection of quantum mechanics to the probability theory. In last years, quantum mechanics became an interesting topic for computer scientists which try to exploits its power for solving computational problems. A very good textbook covering this attempt is [CNC00].

One interesting branch of the mathematical formalism of quantum mechanics is the field of quantum logic developed by von Neumann [vN32]. Quantum logic is a non-standard logic based on projectors of a complex separable Hilbert space. Many mathematicians investigated the properties of quantum logic. A concise overview over the most important results is given by [Zie05].

Many concepts of information retrieval can be found in the field of linear algebra and probability theory. Rijsbergen as one prominent information retrieval expert describes in his book [vR04] the strong relationship between concepts of quantum mechanics and information retrieval.

In this work we try to use the mathematical concepts behind quantum mechanics and logic as a unifying framework for information retrieval and database query processing. This is motivated for example in [Sch04, SSH05]. The attempt to combine both fields is not new. At the beginning of the nineties techniques of fuzzy logic [Zad88] were applied to traditional database technology in order to cope with vagueness. Much research was done on developing fuzzy-databases with corresponding fuzzy query languages. [Bol94] introduces a fuzzy ER-model together with a calculus language using fuzzy-logic. Other examples are [GMPC98, BP95] which investigate how to develop a fuzzy-SQL language. [Tak93] sketches the design of a fuzzy calculus, fuzzy algebra and a mapping between them. However, this work suffers from an incomplete formalization.

Most extensions of the relational model by imprecision via fuzzy-logic were performed on the relational algebra, see e.g. [ABSS98, CMPT00, SS04]. All the fuzzy-approaches suffer from a weak support of the semantics of membership values. They consider membership values as given and do not evaluate the semantics of underlying query terms.

An interesting approach to combine the information retrieval world with the database world via probability theory was proposed in [FR97]. There, the relational model and the relational algebra were enhanced by the concept of probability. To every tuple an event expression is assigned which allows the computation of probability values. Basic events are assumed to be given as explicit probability values. Complex events are constructed in correspondence to algebra expressions which derived that tuples. In contrast to this work, our approach does not start with given probability values but provides a unifying framework to *compute* probability values for database queries as well as for retrieval queries.

Chapter 3

Quantum Mechanics and its Relation to Probability Theory

This section gives a short introduction to quantum mechanics and its relation to probability theory. After discussing some notational conventions, we briefly present the four postulates of quantum mechanics. A deeper understanding of quantum mechanics and computation can be found in the textbook [CNC00]. Here, we assume the reader being familiar with linear algebra and complex numbers.

Quantum mechanics deals with vectors of a complex separable Hilbert space **H**. The Dirac notation provides an elegant means to formulate basic concepts of quantum mechanics:

- A so-called *ket* represents a complex column vector. '$|x\rangle$' denotes the ket vector `x`.
- The complex conjugate and transpose (or short the *adjoint*) of a ket is a so-called *bra* (and vice versa). It represents a complex row vector and is denoted by '$\langle x|$'.
- The *inner product* between two kets $|x\rangle$ and $|y\rangle$ returning a scalar equals the matrix product of the adjoint of $|x\rangle$ with $|y\rangle$. It is denoted by the *bracket* '$\langle x|y\rangle$'.
- The *outer product* between two kets $|x\rangle$ and $|y\rangle$ is the matrix product of $|x\rangle$ with the adjoint of $|y\rangle$ and is denoted by '$|x\rangle\langle y|$'. It generates a linear operator.
- The *tensor product* between two kets $|x\rangle$ and $|y\rangle$ is denoted by $|x\rangle \otimes |y\rangle$ or short by $|xy\rangle$. If $|x\rangle$ is `m`-dimensional and $|y\rangle$ `n`-dimensional then $|xy\rangle$ is an `mn`-dimensional ket vector. For example, the tensor product of two 2-dimensional kets $|x\rangle$ and $|y\rangle$ provides:

$$|xy\rangle = |x\rangle \otimes |y\rangle = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \otimes \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 y_1 \\ x_1 y_2 \\ x_2 y_1 \\ x_2 y_2 \end{pmatrix}.$$

  In this paper, we use the following law on the tensor product:

$$\langle xy|ab\rangle = \langle x|a\rangle\langle y|b\rangle.$$

The *norm* of a ket vector $|x\rangle$ is defined as the square root of the inner product of $|x\rangle$ with itself ($\sqrt{\langle x|x\rangle}$). Next, we sketch very informally the four postulates of quantum mechanics.

Postulate 1: Every closed physical system corresponds to a separable complex Hilbert space and every state of the system can be completely described by a normalized (the norm equals one) ket vector of that space.

A generalized form of expressing a state provides the *density operator*. It allows the formulation of some kind of uncertainty if we do not know the exact state but having instead an ensemble of possible states $|\varphi_i\rangle$ with their respective probabilities p_i. The density operator is defined by $\rho = \sum_i p_i |\varphi_i\rangle\langle\varphi_i|$. Every density operator is positive[1] and must fulfill the trace condition $tr(\rho) = 1$[2].

Postulate 2: Every evolution of a state $|\varphi\rangle$ of a closed system can be represented by the multiplication of $|\varphi\rangle$ with a unitary operator (matrix) U, that is, $U^\dagger U = UU^\dagger = I$ must hold[3]. The new state $|\varphi'\rangle$ is given by $|\varphi'\rangle = U|\varphi\rangle$. It can be easily shown that a unitary operator cannot change the norm.

If a state is described by a density operator ρ then the transformed density operator is given by $\rho' = U\rho U^\dagger$.

Postulate 3: This postulate describes the measurement of a state which enables us to compute the probabilities of different outcomes. If a certain outcome is measured then the system is automatically changed to the corresponding state. Here, we focus just on the simplified measurement by using projectors (each represents one possible outcome) and ignore effects on the state. A *projector* $p = \sum_i |i\rangle\langle i|$ is a self-adjoint ($p = p^\dagger$), idempotent ($pp = p$), linear, and continuous operator $p : \mathbf{H} \to \mathbf{H}$ defined over a set of orthonormal vectors $|i\rangle$. It performs a projection of a given vector onto a respective vector subspace. The probability of an outcome corresponding to a projector p and a state ρ is defined by calculating $tr(p\rho)$. Very important for the intuitive understanding is the following connection to the inner product for $\rho = |\varphi\rangle\langle\varphi|$:

$$\begin{aligned} tr(p\rho) &= tr(\sum_i |i\rangle\langle i|\rho) = \sum_i tr(|i\rangle\langle i||\varphi\rangle\langle\varphi|) \\ &= \sum_i tr(\langle i|\varphi\rangle\langle\varphi|i\rangle) = \sum_i \langle i|\varphi\rangle\langle\varphi|i\rangle. \end{aligned}$$

Thus, the probability equals the squared length of the state vector $|\varphi\rangle$ after its projection onto the subspace spanned by the vectors $|i\rangle$.

Postulate 4: This postulate defines the composition of various closed systems to one system. The state of a composed system is constructed by applying the tensor product '$\otimes$' on the subsystem states. The vectors of a measurement are constructed analogously.

Figure 3.1 illustrates the connection between quantum mechanics and probability theory for the real, 2-dimensional space. Please notice, that the base vectors $|0\rangle$ and $|1\rangle$ are orthonormal. The measurement of the density operator $|\varphi\rangle\langle\varphi|$ by the projector $|0\rangle\langle 0|$ provides the squared portion of $|\varphi\rangle$ on the base vector $|0\rangle$ which equals a^2. Analogously, the projector $|1\rangle\langle 1|$ provides b^2. Due to Pythagoras and the normalization of $|\varphi\rangle$ both values sum up to one. In quantum

[1]Positivity means $\forall|\varphi\rangle : \langle\varphi|\rho|\varphi\rangle \geq 0$.

[2]'$tr(\rho)$' denotes the trace of matrix ρ and equals the sum of its diagonal elements.

[3]The symbol '$\dagger$' denotes the adjoint of a matrix and 'I' denotes the identity matrix.

mechanics, where $|0\rangle\langle 0|$ and $|1\rangle\langle 1|$ represent two possible outcomes of a measurement, the values a^2 and b^2 give the probabilities of the respective outcomes. Since both outcomes are independent ($\langle 0|1\rangle = 0$) and complete ($|0\rangle\langle 0| + |1\rangle\langle 1| = I$) they sum up to one.

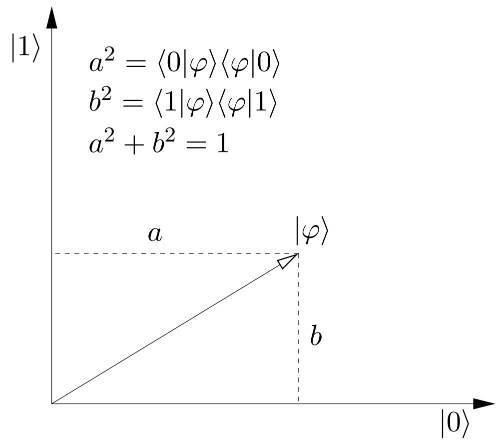

Figure 3.1: Pythagoras and probabilities

Following [vR04] we will discuss two aspects which explain why quantum mechanics may serve as an appropriate model for information retrieval:

1. Information retrieval means to estimate the probability that a database object is relevant to a given query object. One frequently used retrieval model is the real vector space model where the scalar product is utilized to estimate probability values.

2. Quantum mechanics provides a framework for unifying the notion of geometry, probability, and logic[4]. Gleason's theorem [Gle57] establishes the connection between probability theory and geometry. First, we define a probability measure where the join of projectors is denoted by '$\vee$', **H** is a Hilbert space, and p is a projector defining a subspace.

   Definition 3.1 (probability measure) *A (countably additive) probability measure on* $L(\boldsymbol{H})$ *is a mapping* $\mu : L \to [0,1]$ *such that* $\mu(\boldsymbol{I}) = 1$ *and, for any sequence of pair-wise orthogonal projectors* $p_i, i = 1, 2, \ldots : \mu(\vee_i p_i) = \sum_i \mu(p_i)$.

   Theorem 3.1 (Gleason) *Let* $\boldsymbol{H}$ *be a Hilbert space having dimension* > 2. *Then every countably additive probability measure on* $L(\boldsymbol{H})$ *has the form* $\mu(p) = tr(\rho p)$ *for a density operator* ρ *on* $\boldsymbol{H}$.

Related to the problem of information retrieval are classification problems in the field of pattern recognition. Especially, in support vector machines the classification is based on computing the scalar product of samples with a hyperplane representing a subspace. Furthermore, the kernel theory [SS02] gives us a way to deal linearly with non-linearly separable classification problems by mapping them into a high-dimensional vector space. Kernel PCA can be used to extract

[4]Quantum logic will be introduced in next section.

feature values and to embed them in a Hilbert space. These aspects together demonstrate the universality of the Hilbert space model.

The main idea of supporting information retrieval in quantum mechanics is to model database objects as density operators and queries as projectors within a well-designed Hilbert space. A density operator as database object encapsulates all the possible results of potential measurements whereas the projectors define subspaces. Together they form a probability measure.

Chapter 4

Quantum Logic

Following [Zie05], we develop the main concepts of quantum logic originally developed by von Neumann [vN32]. The starting point is the set P of all projectors of a complex separable Hilbert space $\mathbf{H}$ of dimensions greater than two. Each projector $p \in P$ is bijectively related to a closed subspace via $p(\mathbf{H})$. The subset relation $p_1(\mathbf{H}) \subseteq p_2(\mathbf{H})$ on P which is equivalent to $p_2 p_1 = p_1 p_2 = p_1$[1] forms a complete poset. Furthermore, we obtain a lattice with the following binary operations `meet` ($\wedge$) and `join` ($\vee$) being defined as

$$p_1 \wedge p_2 \equiv p_{p_1(\mathbf{H}) \cap p_2(\mathbf{H})} \qquad p_1 \vee p_2 \equiv p_{closure(p_1(\mathbf{H}) + p_2(\mathbf{H}))}.$$

Thus, the laws of commutativity, associativity, and absorption are fulfilled.

Quantum logic does not constitute a Boolean logic since the distribution law is violated. For example, if $|x\rangle$ and $|y\rangle$ are two mutually orthonormal ket vectors then we can define three projectors: $p_1 = |x\rangle\langle x|$, $p_2 = |y\rangle\langle y|$, and $p_3 = |v\rangle\langle v|$ where $|v\rangle = 1/\sqrt{2}(|x\rangle + |y\rangle)$. Then we have

$$p_3 \wedge (p_1 \vee p_2) = p_3 <> 0 = 0 \vee 0 = (p_3 \wedge p_1) \vee (p_3 \wedge p_2).$$

The orthocomplement (negation) for our quantum logic is defined as $\neg p \equiv I - p$ encompassing all orthogonal projectors. Orthogonality between two projectors is symmetric and is defined by $p_1 p_2 = p_2 p_1 = 0$ or equivalently by $p_1(I - p_2) = p_1$. As consequence we obtain an ortholattice fulfilling

1. *compatibility:* $p_1(\mathbf{H}) \subseteq p_2(\mathbf{H}) \;=: \neg p_2(\mathbf{H}) \subseteq \neg p_1(\mathbf{H})$ and
2. *invertibility:* $p \vee \neg p = I, p \wedge \neg p = 0,$ and $\neg\neg p = p$.

From these laws the de Morgan laws can be concluded.

The ortholattice of projectors fulfills furthermore the law of orthomodularity:

$$p_1(\mathbf{H}) \subseteq p_2(\mathbf{H}) \;=: p_1 \vee (\neg p_1 \wedge p_2) = p_2.$$

Therefore, we obtain an orthomodular lattice of projectors.

In this paper, we have to embed the Boolean logic of the relational calculus into quantum logic. However, the violation of the distribution law seems to be a serious problem. In fact, this is not really a problem if we introduce the concept of commuting projectors.

[1] The projection of subspace p_1 onto a containing subspace p_2 equals p_1.

Definition 4.1 (Commuting projectors) *Two projectors p_1 and p_2 of a Hilbert space $\boldsymbol{H}$ are called* commuting projectors *iff $p_1 p_2 = p_2 p_1$ holds.*

From linear algebra we know that two projectors $p_1 = \sum_i |i\rangle\langle i|$ and $p_2 = \sum_j |j\rangle\langle j|$ commute if and only if the ket vectors $|i\rangle$ and $|j\rangle$ are basis vectors of the same orthonormal basis of the Hilbert space. In that case, we can write $p_1 = \sum_{i_1} |k_{i_1}\rangle\langle k_{i_1}|$ and $p_2 = \sum_{i_2} |k_{i_2}\rangle\langle k_{i_2}|$ where the ket vectors $|k\rangle$ form an orthonormal basis. If two projectors commute then their join corresponds to the union of the respective ket vectors and their meet to the intersection. Thus, all projectors over one given orthonormal basis form a Boolean logic. This is generalized by the following theorem.

Theorem 4.1 (Foulis-Holland) *Let L be an orthomodular lattice and a, b, c in L such that any one of them commutes with the other two. In this particular case the distributivity law holds.*

The following quotation from [Mar77] summarizes the main idea of the quantum theory:

> '*Quantum theory is simply the replacement in standard probability theory of event-as-subset-of-a-set (abelian, distributive) by event-as-subspace-of-a-Hilbert-space (non-abelian, non-distributive).*'

Chapter 5

Database Retrieval

In this section we develop basic concepts for mapping database tuples and basic relational calculus queries to quantum mechanics and quantum logic. We restrict our discussion to select-conditions examining the equality of a database attribute value and a given constant. Processing such select-conditions within quantum mechanics means to produce values from $[0, 1]$. For a correct simulation of the calculus semantics only a value of one is regarded as `true`.

First, we map database tuples into states of a Hilbert space. For simplicity, we assume a one-attribute-tuple containing a non-negative numerical a is given. Please remember, that state vectors are normalized. Therefore, we cannot directly map an attribute value to a one-dimensional ket vector. Due to the restriction of normalization and the demand for one degree of freedom to model a value a we map it to a two-dimensional ket vector $|a\rangle$:

$$a \mapsto |a\rangle = \frac{1}{\sqrt{a^2+1}} \begin{pmatrix} 1 \\ a \end{pmatrix}$$

Thus, the value a is encoded as the normalized ratio of a data-dimension value to a reference-dimension value. The corresponding density operator is given by $\rho_a = |a\rangle\langle a|$. Inversely, the hidden attribute value a can be extracted from a given density operator ρ_a by

$$\rho_a = |a\rangle\langle a| \mapsto a = \sqrt{\frac{tr(|1\rangle\langle 1|\rho_a)}{tr(|0\rangle\langle 0|\rho_a)}} \text{ where } |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

A constant 'c' of a select-condition is analogously mapped to the projector $p_c = |c\rangle\langle c|$ with

$$c \mapsto |c\rangle = \frac{1}{\sqrt{c^2+1}} \begin{pmatrix} 1 \\ c \end{pmatrix}.$$

Since the ket vectors $|a\rangle$ and $|c\rangle$ both are two-dimensional we call them in conformity with the terms of quantum computation *qubits*.

Computing the degree of quantum matching between an attribute value a and a select-condition with the constant c provides the squared cosine of the angle between $|a\rangle$ and $|c\rangle$

$$tr(p_c\rho_a) = \langle a|c\rangle\langle c|a\rangle = \frac{(1+ac)^2}{(a^2+1)(c^2+1)}. \tag{5.1}$$

This measurement produces a value from the range $[0, 1]$. From the formula one can immediately see that the case $a = c$ produces the value one. Furthermore, we obtain a value near to zero (orthogonality in the geometric interpretation) only if one value is very high whereas the other value equals zero[1]. Figure 5.1 depicts the graph for probability values which we obtain from formula 5.1 if we compare two values.

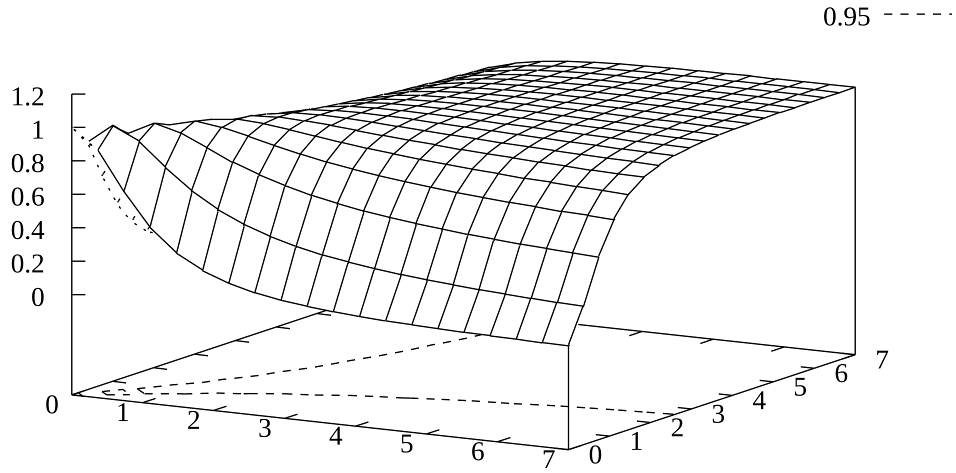

Figure 5.1: Graph for $\frac{(1+ac)^2}{(a^2+1)(c^2+1)}$

As explained in Section 3, the result of a measurement can be interpreted as a probability value. Of course, the measured value in our scenario can heavily be connected to the probability of a meaningful event. Therefore, we apply a user-defined bijection $f : dom(A) \to [0, \infty]$ on attribute values before quantum encoding is performed. This gives the user a means to assign some semantics to the resulting probability values. The mapping function should respect the order relation on the attribute values. The encoding $\rho_a = |a\rangle\langle a|$ and $p_c = |c\rangle\langle c|$ of an attribute value a and a select-condition with the constant c are thus refined by

$$a \mapsto |a\rangle = \frac{1}{\sqrt{f(a)^2 + 1}} \begin{pmatrix} 1 \\ f(a) \end{pmatrix} \qquad c \mapsto |c\rangle = \frac{1}{\sqrt{f(c)^2 + 1}} \begin{pmatrix} 1 \\ f(c) \end{pmatrix} \tag{5.2}$$

whereas the decoding is given by

$$\rho_a = |a\rangle\langle a| \mapsto a = f^{-1}\left(\sqrt{\frac{tr(|1\rangle\langle 1|\rho_a)}{tr(|0\rangle\langle 0|\rho_a)}}\right). \tag{5.3}$$

Such a mapping enables us to encode non-numerical attributes, too.

Example 5.1 (Attribute value mapping) *Assume a database value is restricted to one of the eight characters 'A' to 'H'. First, these characters are bijectively mapped to the integers 0 to 7 ('A'* $\to 0, \ldots,$ *'H'*$\to 7$*). Second, we map those integers to density operators producing a dependence of measured results from the distance* $d = |a - c|$ *representing the absolute error. Such*

[1]Please remember, that we mapped non-negative, real numericals only.

a symmetry is achieved by linearly mapping the eight values to angles by applying the function $f : a \mapsto \tan a\pi/16$. This produces a measurement value given by $\cos^2 d\pi/16$. Figure 5.2 depicts the geometry of that mapping. The corresponding measurement graph is given in Figure 5.3.

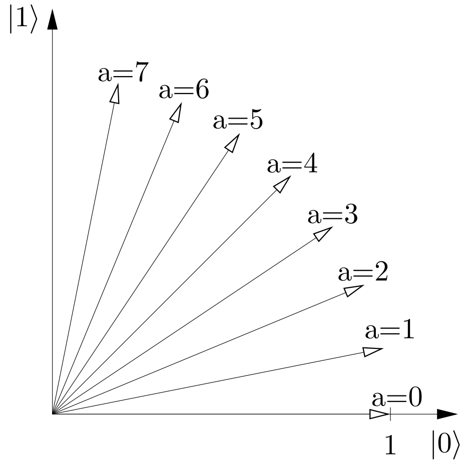

Figure 5.2: Equi-angular mapping of a

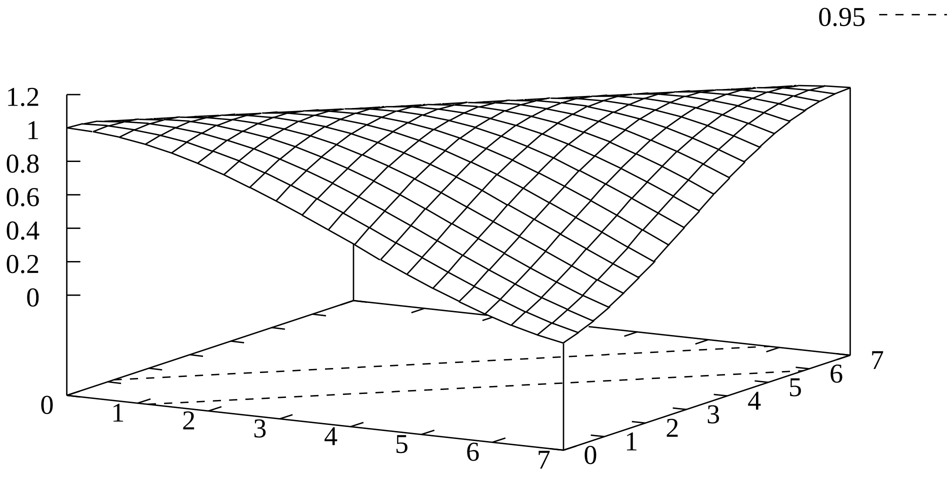

Figure 5.3: Graph for $\frac{(1+ac)^2}{(a^2+1)(c^2+1)}$

A typical database tuple contains more than one attribute. Therefore, we have to adapt our mapping to the multi-attribute case. A multi-attribute tuple can be regarded as a composite quantum system. Applying postulate 4, we use the tensor product for constructing multi-attribute density operators.

Assume, a database tuple $t = (a_1, \ldots, a_n)$ contains n attribute values and $|a_1\rangle, \ldots, |a_n\rangle$ are the corresponding qubits. Then, the ket vector

$$|t\rangle = |a_1\rangle \otimes \ldots \otimes |a_n\rangle = |a_1..a_n\rangle$$

represents the whole tuple. The corresponding density operator is given by $\rho_t = |t\rangle\langle t|$.

A single-attribute select-condition $A_j = c$ on a multi-attribute-tuple must be prepared accordingly. However, the ket vector $|c\rangle$ needs to be combined with all orthonormal basis vectors of the remaining qubits producing the projector p_c.

$$p_c = \sum_{i_k \in \{0,1\}} |i_1..i_{j-1} c\, i_{j+1}..i_n\rangle\langle i_1..i_{j-1} c\, i_{j+1}..i_n|.$$

Computing the measurement result provides

$$\begin{aligned} tr(p_c\rho_t) &= \sum_{i_k \in \{0,1\}} \langle a_1..a_n|i_1..i_{j-1} c\, i_{j+1}..i_n\rangle\langle i_1..i_{j-1} c\, i_{j+1}..i_n|a_1..a_n\rangle \\ &= \sum_{i_k \in \{0,1\}} \begin{array}{c} \langle a_1|i_1\rangle..\langle a_{j-1}|i_{j-1}\rangle\langle a_j|c\rangle\langle a_{j+1}|i_{j+1}\rangle..\langle a_n|i_n\rangle * \\ \langle i_1|a_1\rangle..\langle i_{j-1}|a_{j-1}\rangle\langle c|a_j\rangle\langle i_{j+1}|a_{j+1}\rangle..\langle i_n|a_n\rangle \end{array} \end{aligned}$$

Every normalized qubit $|x\rangle$ combined via the inner product with all orthonormal basis vectors $|i\rangle$ provides the value 1: $\sum_{i \in \{0,1\}} \langle x|i\rangle\langle i|x\rangle = 1$. Thus, we can simplify the last formula to

$$tr(p_c\rho_t) = \langle a_j|c\rangle\langle c|a_j\rangle = \frac{(1 + a_j c)^2}{(a_j^2 + 1)(c^2 + 1)}. \tag{5.4}$$

The formula equals the measurement Formula 5.1 where we considered just one qubit. That is, the measured value for a one-attribute select-condition is completely independent from the existence of other attributes.

A complex query condition in the relational calculus is constructed by recursively applying conjunction, disjunction and negation on atomic conditions. So far, we explained how to map single-attribute select-conditions to projectors. As we will demonstrate now, disjunction, conjunction and negation on select-conditions have their counterparts in quantum logic. That is, for combining two projectors conjunctively we apply the `meet` operator returning a new projector. Analogously, disjunction corresponds to the `join` operator and the negation of a condition is reflected by the negation of a projector. Quantum logic behaves like Boolean logic if the projectors commute. Furthermore, it allows the interpretation of measurement results as probability values.

Negation: Assume, a select-condition with constant c on attribute A_1 is given. Then, the corresponding projector is defined by:

$$p_c = \sum_{i_k \in \{0,1\}} |c\, i_2..i_n\rangle\langle c\, i_2..i_n|$$

Its negation provides $p_{\neg c} = I - p_c$. For a given density operator ρ_t, the measurement with the negated projector yields:

$$tr(p_{\neg c}\rho_t) = tr((I - p_c)\rho_t) = tr(\rho_t) - tr(p_c\rho_t) = 1 - tr(p_c\rho_t). \tag{5.5}$$

Here, we used the trace condition of the density operator.

Conjunction: Assume, two select-conditions $A_1 = c$ and $A_2 = d$ are given. The corresponding projectors compute as

$$\begin{aligned} p_c &= \sum_{i_k \in \{0,1\}} |c\, i_2..i_n\rangle\langle c\, i_2..i_n| \text{ and} \\ p_d &= \sum_{i_k \in \{0,1\}} |i_1 d\, i_3..i_n\rangle\langle i_1 d\, i_3..i_n|. \end{aligned}$$

Essential is here the question whether both projectors commute. They commute if they can be defined on the same orthonormal basis. Obviously, the vectors $|0\rangle$ and $|1\rangle$ of the qubits 3 to n are basis vectors of the same orthonormal basis. But do $|c\rangle\langle c|$ and $q_1 = |0\rangle\langle 0| + |1\rangle\langle 1|$ commute, too? Since q_1 equals the identity matrix of that qubit we obtain $|c\rangle\langle c|I = |c\rangle\langle c| = I|c\rangle\langle c|$. That is, both one-qubit-projectors commute. The same holds for the second qubit $|d\rangle\langle d|I = |d\rangle\langle d| = I|d\rangle\langle d|$. From this discussion we see, that the projectors p_c and p_d can be formulated on the same orthonormal basis and do, therefore, commute. Intersecting the respective sets of eigenvectors of both projectors provides

$$p_{c\wedge d} = \sum_{i_k \in \{0,1\}} |c\, d\, i_3.._n\rangle\langle c\, d\, i_3..i_n|.$$

Computing the measurement with respect to a density operator $\rho_t = (a_1..a_n)$ yields

$$\begin{aligned} tr(p_{c\wedge d}\rho_t) &= \langle a_1 a_2|c\, d\rangle\langle c\, d|a_1 a_2\rangle = \langle a_1|c\rangle\langle c|a_1\rangle\langle a_2|d\rangle\langle d|a_2\rangle && (5.6) \\ &= tr(p_c\rho_t)tr(p_d\rho_t). && (5.7) \end{aligned}$$

Thus, the measured results for conjunctively combined commuting projectors are multiplied. This conforms the probabilistic theory for independent events.

Disjunction: From Section 4 we know that quantum logic respects the de Morgan law. Therefore, we can compute the disjunction of projectors over conjunction and negation and obtain:

$$\begin{aligned} tr(p_{c\vee d}\rho_t) &= 1 - (1 - tr(p_c\rho_t))(1 - tr(p_d\rho_t)) && (5.8) \\ &= tr(p_c\rho_t) + tr(p_d\rho_t) - tr(p_{c\wedge d}\rho_t). && (5.9) \end{aligned}$$

The discussed semantics of conjunction, disjunction, and negation obey the rules of probability theory for independent events. Furthermore, the logical operations on non-overlapping projectors equal the algebraic product and the algebraic sum being a t-norm and a t-conorm of fuzzy-logic [Zad88], respectively. However, our theory gives us more semantics about the underlying conditions: The formulas 5.7 and 5.9 are valid on non-overlapping conditions only. The problem of violated idempotence of the algebraic product does not exist in our theory since underlying projectors would not be non-overlapping. Next, we discuss overlapping projectors. Two projectors with overlapping select-attributes commute only if the overlapping select-conditions are equal. Assume, projectors p_1 and p_2 are given which express the same select-condition c on attribute A and contain no further select-condition. Both projectors commute because they are defined on the same orthonormal basis. Therefore, the `meet` operation (intersection of the eigenvector sets) and the `join` operator (union of the eigenvector sets) on p_1 and p_2 return $p_1 = p_2$ fulfilling idempotence. As consequence, the conjunction of two projectors provides a measurement where the measured results of single-attribute conditions on non-overlapping attributes are multiplied whereas redundant attribute-conditions are considered only once.

Non-commuting projectors from select-conditions can only occur if different select-conditions on one attribute are defined. In this case, the conjunction refers to the intersection of the corresponding subspaces which yields the empty set. The disjunction, however, finds a projector for a space which is spanned by two different ket vectors. That is, such a disjunction returns the identity matrix for that qubit. Both semantics do not correctly reflect the semantics of database queries.

Now, we consider select-conditions which require equivalence of different attributes. We assume a two-value-tuple (x, y) is given. Its representation in the Hilbert-space is

$$|xy\rangle = \frac{1}{\sqrt{x^2+1}} \begin{pmatrix} 1 \\ x \end{pmatrix} \otimes \frac{1}{\sqrt{y^2+1}} \begin{pmatrix} 1 \\ y \end{pmatrix} = \frac{1}{\sqrt{x^2+1}\sqrt{y^2+1}} \begin{pmatrix} 1 \\ y \\ x \\ xy \end{pmatrix}.$$

In order to construct a projector for the equality-condition we have to investigate the space spanned by arbitrary $|xy\rangle$ vectors where $x = y$ holds. It can be immediately seen that $x = y$ is true if and only if the values of the second and the third dimension are equal. Thus, the vector space for the equality-condition is spanned by the orthonormal vectors $|00\rangle, 1/\sqrt{2}(|01\rangle + |10\rangle)$, and $|11\rangle$ where

$$|00\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) = \begin{pmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix} \qquad |11\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

hold. Please observe, that $1/\sqrt{2}(|01\rangle + |10\rangle)$ cannot be expressed as tensor product of any set of subsystem vectors. Therefore, it is called an *entangled state*. The corresponding projector ρ_{xx} is given by

$$p_{xx} = |00\rangle\langle 00| + \frac{(|01\rangle + |10\rangle)(\langle 01| + \langle 10|)}{2} + |11\rangle\langle 11| = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Measuring a density operator $\rho_{xy} = |xy\rangle\langle xy|$ yields

$$tr(p_{xx}\rho_{xy}) = \frac{x^2y^2 + \frac{(x+y)^2}{2} + 1}{(x^2+1)(y^2+1)} = \frac{x^2y^2 + \frac{(x+y)^2}{2} + 1}{x^2y^2 + x^2 + y^2 + 1}.$$

The result equals the maximum one if and only if x equals y. Otherwise, we obtain a value smaller than one but greater than 0.5. The function converges the minimum of 0.5 if one value is zero whereas the other one grows towards infinity. Figure 5.4 depicts the graph of the measurement formula. The isolines of the figure unveils that if the values x and y both are no small values then the formula returns a value very near to one. This effect is caused by the irregular encoding of values to angles. By choosing an appropriate bijection for the encoding we can overcome that problem. Figure 5.5 depicts the graph where we applied the encoding as described in Example 5.1.

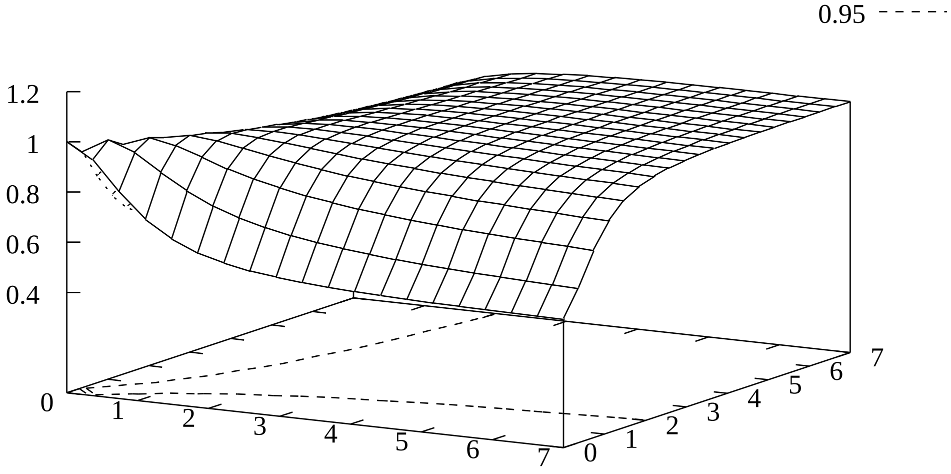

Figure 5.4: Graph for $\frac{x^2y^2+\frac{(x+y)^2}{2}+1}{(x^2+1)(y^2+1)}$

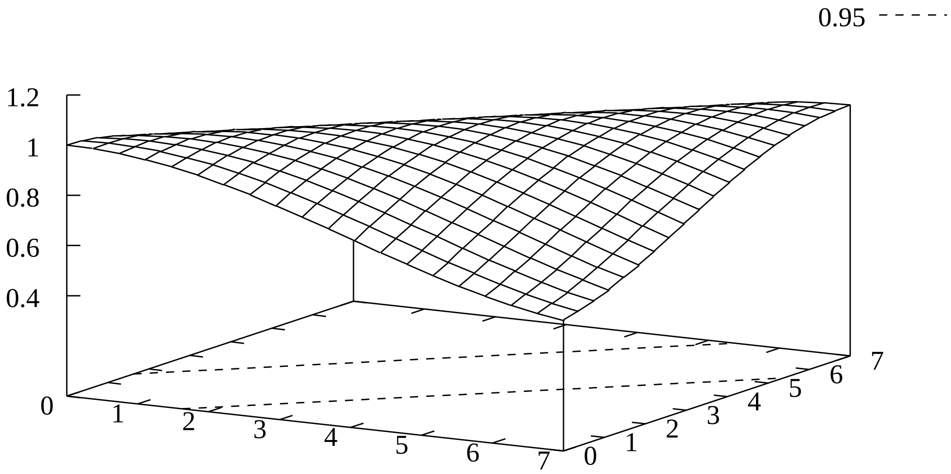

Figure 5.5: Graph for equi-angular value

Next, we examine the consistency between equality-conditions and constant-select-conditions by combining them conjunctively. Let the following projectors on two qubits be given: $P_{A_1=A_2}$ for condition $A_1 = A_2$, $P_{A_1=c}$ for $A_1 = c$, and $P_{A_2=c}$ for condition $A_2 = c$. One essential consistency requirement for the equality-condition can be formulated as

$$P_{A_1=A_2 \wedge A_1=c} = P_{A_1=A_2 \wedge A_2=c} = P_{A_1=c \wedge A_2=c}. \tag{5.10}$$

Not surprisingly, the equal-condition projector $P_{A_1=A_2}$ does not commute with the select-condition projectors $P_{A_1=c}$ and $P_{A_2=c}$. As already mentioned, the vector $1/\sqrt{2}(|01\rangle + |10\rangle)$ is an entangled state which cannot be expressed by the tensor product of any subsystem basis vectors.

Combining $P_{A_1=A_2}$ and $P_{A_1=c}$ conjunctively yields

$$P_{A_1=A_2} \wedge P_{A_1=c} = P_{A_1=A_2 \wedge A_1=c} = |cc\rangle\langle cc|.$$

In order to prove this proposition please notice that $|cc\rangle$ is a vector within the space of $P_{A_1=A_2}$ and $P_{A_1=c}$ that is $P_{A_1=A_2}|cc\rangle = P_{A_1=c}|cc\rangle = |cc\rangle$. Additionally, we will show, that $P_{A_1=A_2 \wedge A_1=c}$ is composed of exactly one eigenvector. This can be proven by applying the deMorgan-law. Negating $P_{A_1=A_2}$ provides $P_{\neg(A_1=A_2)} = 1/2(|01\rangle - |10\rangle)(\langle 01| - \langle 10|)$. Negating $P_{A_1=c}$ yields $P_{\neg(A_1=c)} = |\overline{c}0\rangle\langle\overline{c}0| + |\overline{c}1\rangle\langle\overline{c}1|$ where

$$|\overline{c}\rangle = 1/\sqrt{c^2+1}(-c\ 1)^\dagger.$$

The vectors $|\overline{c}0\rangle$ and $|\overline{c}1\rangle$ are mutually orthogonal and span a two-dimensional space. Since, furthermore, $P_{\neg(A_1=A_2)}$ and $P_{\neg(A_1=c)}$ do not commute the vector space of $P_{\neg(A_1=A_2)\vee\neg(A_1=c)}$ is three-dimensional and, henceforth, its negation one-dimensional.

Analogously, the conjunction of $P_{A_1=A_2}$ and $P_{A_2=c}$ yields $P_{A_1=A_2 \wedge A_2=c} = |cc\rangle\langle cc|$ fulfilling our consistency requirement.

Chapter 6

Conclusion and Outlook

Quantum mechanics and quantum logic together provide an elegant theory combining concepts from geometry (Hilbert space), logic, and probability theory. Queries of some applications involve database queries as well as retrieval queries. Therefore, the main idea of this work is to use quantum mechanics and logic as a unifying theory for information retrieval and database querying.

After introducing quantum mechanics and quantum logic we investigated the potential of supporting database queries. Here, we focused on select-conditions performing a constant-comparison with database tuples and equality-conditions. Such database queries can be successfully simulated by quantum mechanics. Their connection to retrieval queries is straightforward since the involved values of a retrieval query are usually disjoint from database attributes. Thus, the projectors of both query types commute.

The measurements results of our quantum retrieval can be regarded as probability values. The conjunction, disjunction and negation conforms the semantics of the probability theory. In this way, we introduced a notion of proximity into database queries. Furthermore, the paper gives us some interesting insights into the problem of relating logic to probability theory.

The paper describes mainly theoretical results. However, in order to use the results practically one can simply implement the formulas 5.2, 5.3, 5.4, 5.5, 5.7, 5.9 and 5.10. In this way, we do not have to implement the concepts of quantum mechanics and quantum logic but we can directly use its results.

The paper did not map all concepts of the relation calculus to quantum mechanics. So far, one open problem seems to be the restriction to commuting projectors. In future, we will work on following aspects:

- different conditions (not commuting) on one attribute and
- range queries.

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