SOME EQUATION-LESS CONSTRUCTIONS IN DOUBLE CATEGORIES.

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ABSTRACT. Some new constructions in the theory of double categories are taken into account which can be defined without equations. Instead, we can define a relation which plays the role of the equality relation in the usual category theory. Many well-known features like determinacy up-to isomorphism and others are also valid for the new equation-less constructions.
Herein, double categories are used in the same way as in the GADDUCCIs and MONTANARIs 'tile model' ([GM96]) for the description of the dynamics of process algebras. The equation-less definition of double-categorical constructions are more adequate because equations are as able to express dynamics as the second dimension of the double-categorical structure if one underlies a respective semantics as in MESEGUERs 'rewrite logic' ([Mes92]).
Besides this one can construct a product in the category of partial functions which corresponds to the set-theoretical product by using the new construction.

Contents

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List of Figures

List of Tables

1. Motivation

1.1. EXAMPLE. The computable function of addition of two natural numbers is to be specified algebraically. This function can be defined by total equational logic.
Initial-algebraically specifying of this adder means to factorize a term set $T_{\mathbb{N}}$. This term set is the smallest set with respect to set inclusion which contains

- a term 0,
- for each term $x \in T_{\mathbb{N}}$ a term $\text{succ}\,(x) \in T_{\mathbb{N}}$ and
- for every two terms $x, y \in T_{\mathbb{N}}$ a term $x + y \in T_{\mathbb{N}}$.

the equivalence relation for the factorization is defined by the following equations:

$$0 + y \;=\; y \quad \text{for each } y \in T_{\mathbb{N}}. \tag{1}$$

$$\mathrm{succ}\,(x) + y \;=\; \mathrm{succ}\,(x+y) \quad \text{for each } x, y \in T_{\mathbb{N}}. \tag{2}$$

1.2. THE DYNAMICS OF INITIAL-ALGEBRAIC SPECIFICATIONS. How does this computable function get its operational semantics? The functions which one can specify by an initial model of partial equational logic (cf. for instance [Rei87]) are known to be just all the partially computable functions. However, *functions* do not have any operational semantics or behavior. But the *process* of computing is the *theorem prover* of the logic which is used for specification, i. e. in this case the equational logic. For instance, this demonstrates how the *process* which adds 3 and 2 corresponds to a logical derivation of a theorem by stepwise application of the above equations in the previous example:

$$\dfrac{\dfrac{\dfrac{0+2=2 \qquad :\mathbb{N}}{1+2 \;=\; 3 \qquad :\mathbb{N}}}{2+2 \;=\; 4 \qquad :\mathbb{N}}}{3+2 \;=\; 5 \!:\mathbb{N}}$$

1.3. ON THE USAGE OF DOUBLE CATEGORIES. As one can see, not only the specification logic plays a role but also the *inference relation*. The question would be interesting whether a *two-dimensional structure* like those of double categories can consider this.

1.4. EQUATIONAL DEFINITION OF CATEGORICAL CONSTRUCTIONS. Specifications in the partial equational logic are known to be representable by categories with finite limits in a manner which is comparable with lemma 2.13 (page 6) and LAWVEREs categorical interpretation ([Law63]). Categories as well as many categorical constructions like those finite limits are algebraic structures which can be presented by partial equational logic. In this paper categorical constructions are considered which can be defined without equations but using the two-dimensional structure of double categories. Besides this one can define an substitute for the equational relation using the double category structure which is called 'equimorphism'. These features oblige the usage of double categories as tool for the description of dynamics of process algebras because the equational relation in a specification of a partially computable function can be seen as part of the definition of the operational semantics if one underlies the semantic interpretation of *change* for this relation like in MESEGUERS 'rewrite logic' ([Mes92]).

2. Dynamics of flow graphs

2.1. THE PURPOSE OF THIS SECTION. ...is

- to give a syntactical presentation of the expressions of double category theory which facilitates the imagination of the reader and
- to explain an application field of the theory of double categories, i. e. to describe semantics of data flow graphs in an abstract manner.

2.2. Data. The following definitions introduce the *horizontal partial category* which is one component of a double category. In our application it is interpreted in the Lawveres sense ([Law63]) as category of contexts (objects) and substitutions (morphisms) and, simultaneously, in Lambeks and Scotts sense ([LS86]) as category of logical propositions (objects) and proofs (morphisms).

2.3. Definition. [Data signature.] *A* data signature Σ *is an ordered pair* $(S_\Sigma, \Omega_\Sigma)$ *with a set* S_Σ *of* ground sorts *and a family* Ω_Σ *of pairwisely disjoint* operation symbol *sets* $\{\Omega_{\Sigma,\Gamma,X} | \Gamma \in S_\Sigma^*, X \in S_\Sigma\}$ [1].

2.4. Example. [Directed graphs.] The data signature Σ of directed graphs is:

$$S_\Sigma := \{\text{node}, \text{edge}\} \tag{3}$$
$$\Omega_{\Sigma,\text{edge},\text{node}} := \{\text{begin}, \text{end}\} \tag{4}$$
$$\Omega_{\Sigma,\Gamma,X} := \emptyset \quad \text{for the other } X \in S_\Sigma,\ \Gamma \in S_\Sigma^*. \tag{5}$$

2.5. Definition. [Context.] *Given a data signature* Σ *and a* S_Σ*-ranked set of* variables $U := \{U_X | X \in S_\Sigma\}$. *An expression of the form*

$$\{x_1 : X_1, \ldots, x_n : X_n\} \tag{6}$$

with $n \geq 0$ *and* $X_i \in S_\Sigma$, $x_i \in U_{X_i}$ *for each* $i \in \{1, \ldots, n\}$ *and* $x_i \neq x_j$ *for each* $i, j \in \{1, \ldots, n\}$, $i \neq j$, $X_i = X_j$ *is called a* context *with respect to the data signature* Σ *and the variable family* U.
A context Γ *with respect to a variable family* U *and a context* Δ *with respect to a variable family* V *of the same data signature* Σ *are referred to be* disjoint *iff for each* $X \in S_\Sigma$ *the sets* U_X *and* V_X *are disjoint.*

2.6. Definition. [Term.] *Given a data signature* Σ.
The smallest set $T_{\Sigma,\Gamma,X}$ *of expressions* $(\forall\Gamma)\, t : X$, *which is closed under the generation rules below, with a context* Γ *with respect to* Σ *and an arbitrary variable set family* U *just like those in the preceding definition:*

$$\frac{}{(\forall \{x_1 : X_i, \ldots, x_n : X_n\})\, x_k : X_k} \quad k \in \{1, \ldots, n\}$$

$$\frac{c \in \Omega_{\Sigma,(),X}}{(\forall\Gamma)\, c : X} \qquad \frac{(\forall\Gamma)\, t_1 : X_1 \quad \ldots \quad (\forall\Gamma)\, t_n : X_n \quad f \in \Omega_{\Sigma,X_1 \ldots X_n,X}}{(\forall\Gamma)\, f\,(t_1, \ldots, t_n) : X}$$

with $X \in S_\Sigma$, $n \geq 0$, $X_i \in S_\Sigma$, $x_i \in U_{X_i}$ *for each* $i \in \{1, \ldots, n\}$ *is called the set of* terms *between the contexts* Γ *and the ground sort* $X \in S_\Sigma$.
A term between the context $\{\}$ *and the ground sort* $X \in S_\Sigma$ *is called* ground term *of the ground sort* X.

[1] S_Σ^* denotes the free monoid over S_Σ.

2.7. Definition. [Substitution.] *Given two contexts* Γ *and* Δ *with respect to a signature* Σ.
A map family

$$\sigma \quad = \quad (\sigma_X \in T_{\Sigma,\Delta,X} \rightarrow T_{\Sigma,\Gamma,X} | X \in S_\Sigma) \tag{7}$$

is called substitution *between them if*

$$\frac{}{\sigma_X\left((\forall\Delta)\, c : X\right) = (\forall\Gamma)\, c : X} \qquad \frac{\forall i \in \{1,\ldots,n\}\,.\sigma_{X_i}\left((\forall\Delta)\, t_i : X_i\right) = (\forall\Gamma)\, t'_i : X_i}{\sigma_X\left((\forall\Delta)\, f\,(t_1,\ldots,t_n) : X\right) = (\forall\Gamma)\, f\,(t'_1,\ldots,t'_2) : X}$$

for each $c \in \Omega_{\Sigma,\{\},X}$ *or* $f \in \Omega_{\Sigma,X_1\ldots X_n,X}$ *with* $n \geq 0$, *resp.*
If $\Gamma = \{\}$ *holds, the substitution is called* ground substitution.

2.8. Notation. *Substitutions between contexts* Γ *and* $\{y_1 : Y_1,\ldots,y_n : Y_n\}$ *are presented as expressions like*

$$(\forall\Gamma)\, y_1 := t_1 : Y_1,\ldots,y_n := t_n : Y_n \tag{8}$$

which denotes that substitution σ *with*

$$\sigma_{Y_i}\left((\forall\,\{y_1 : Y_1,\ldots,y_n : Y_n\})\, y_i : Y_i\right) \quad = \quad (\forall\Gamma)\, t_i : Y_1 \tag{9}$$

for each $i \in \{1,\ldots,n\}$. *That is possible because of this well-known corollary which can be easily proved by structural induction:*

2.9. Corollary. *A substitution* σ *between contexts* Γ *and* $\{y_1 : Y_1,\ldots,y_n : Y_n\}$ *with respect to a signature* Σ *is determined uniquely by the values*

$$\sigma_{Y_i}\left((\forall\,\{y_1 : Y_1,\ldots,y_n : Y_n\})\, y_i : Y_i\right) \quad i \in \{1,\ldots,n\} \tag{10}$$

2.10. Remark. All terms can be regarded as substitutions with $n = 1$.

2.11. Definition. [Renaming.] *Given two variable families* U *and* V *with respect to a data signature* Σ.
A renaming *or an* α-conversion *is a substitution*

$$(\forall\Gamma)\, y_i := x_1 : X_1,\ldots,y_n := x_n : X_n \tag{11}$$

with $n \geq 0$ *and* $X_i, Y_i \in S_\Sigma$ *and* $x_i \in U_{X_i}$ *and* $y_i \in V_{X_i}$ *for each* $i \in \{1,\ldots,n\}$.

2.12. Remark. For each context Γ one can found α-conversions to another context Δ such that Γ and Δ are disjoint. That makes the definitions of the lemma below admissible.

2.13. Lemma. [The horizontal partial category.] *Given a data signature* Σ. *Then a category with finite products is defined as follows: The objects of this category are the contexts with respect to this data signature and the morphisms between two objects* Γ *and* Δ *are the substitutions. Further,*

$$\mathbf{id}_{\Gamma,X}\left((\forall\Gamma)\, t : X\right) \quad = \quad (\forall\Gamma)\, t : X \tag{12}$$

$$(\sigma_X \circ \tau_X)\left((\forall\Gamma)\, t : X\right) \quad = \quad \tau_X\left(\sigma_X\left((\forall\Gamma)\, t : X\right)\right) \tag{13}$$

for each context Γ *and* $X \in S_\Sigma$ *and term* $(\forall\Gamma)\, t : X \in T_{\Sigma,\Gamma,X}$.
The product of the family of contexts which are without loss of generality pairwise disjoint because one can provide pairwise disjointness by α*-conversions which are isomorphisms.*

$$\underline{\Gamma} \;=\; (\Gamma_i | i \in \{1,\ldots,n\}) \tag{14}$$

with $n \geq 0$ *and*

$$\Gamma_i \;:=\; \{x_{i,1} : X_{i,1}, \ldots, x_{i,m_i} : X_{i,m_i}\} \tag{15}$$

with $m_i \geq 0$ *for each* $i \in \{1,\ldots,n\}$ *is*

$$\prod_{i=1}^{n} \Gamma_i \;=\; \{x_{1,1} : X_{1,1}, \ldots, x_{1,m_1} : X_{1,m_1}, \ldots, x_{m,1} : X_{m,1}, \ldots, x_{m,m_1} : X_{m,m_m}\} \tag{16}$$

The morphisms belonging to product are

$$p^k_{\underline{\Gamma},X}\left((\forall\Gamma_i)\, t : X\right) \;=\; \left(\forall \prod_{i=1}^{n} \Gamma_i\right) t : X \tag{17}$$

$$\langle \sigma_1, \ldots, \sigma_n \rangle_{X_{j,k}} \left(\left(\forall \prod_{i=1}^{n} \Gamma_i\right) x_{j,k} : X_{j,k} \right) \;=\; \sigma_{j,X_{j,k}}\left((\forall\Gamma_j)\, x_{j,k} : X_{j,k}\right) \tag{18}$$

where $k \in \{1,\ldots,m_j\}$.

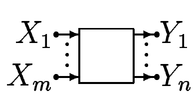
2.14. Flow graphs. Now, that is the turn of a second category, the *vertical partial category* of the double categories in this application.

2.15. Definition. [Agent signature.] *An* agent signature Ψ *is an ordered pair* (S_Ψ, Ξ_Ψ) *with a set* S_Ψ *of* ground sorts *and a family* Ξ_Ψ *of pairwisely disjoint* agent symbol *sets* $\{\Xi_{\Psi,\Gamma,\Delta} | \Gamma, \Delta \in S^*_\Psi\}$.

2.16. Definition. [Flow graphs.] *Given an agent signature* Ψ. *The set* $\mathfrak{mor}_\Psi$ *of the* flow graphs *is the smallest set with respect to set inclusion which is closed under the operations of Tab. 1, factorized by the equations of Tab. 2.*
$\mathfrak{hom}_\Psi(\Gamma, \Delta)$ *with two contexts*

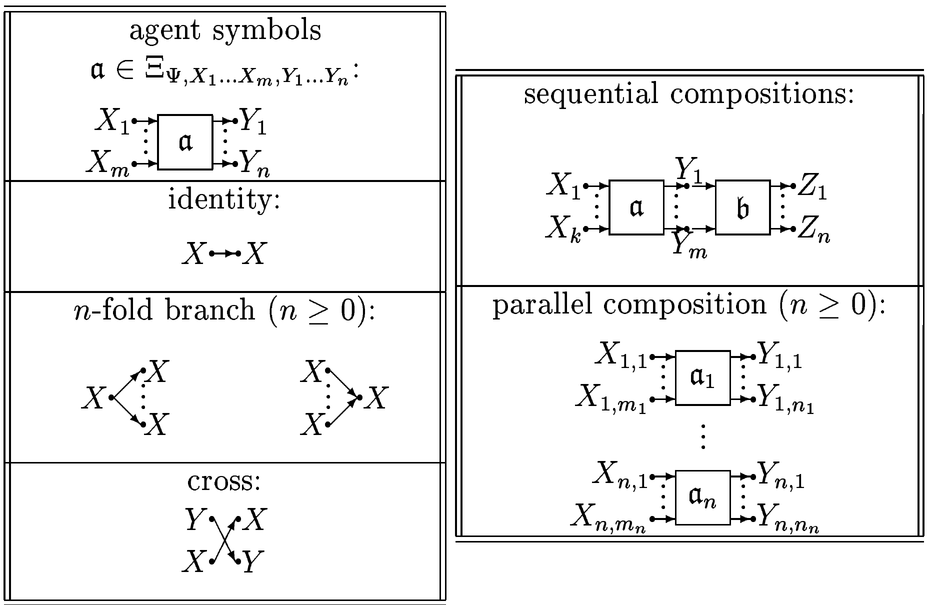
$$\Gamma \;:=\; \{x_1 : X_1, \ldots, x_m : X_m\} \quad m \geq 0 \tag{19}$$
$$\Delta \;:=\; \{y_1 : Y_1, \ldots, y_n : Y_n\} \quad n \geq 0 \tag{20}$$

is the set of the following flow graphs:



(21)

The boxes in Tab. 2 serve as parentheses.



for a given agent signature Ψ

Table 1: Flow graphs

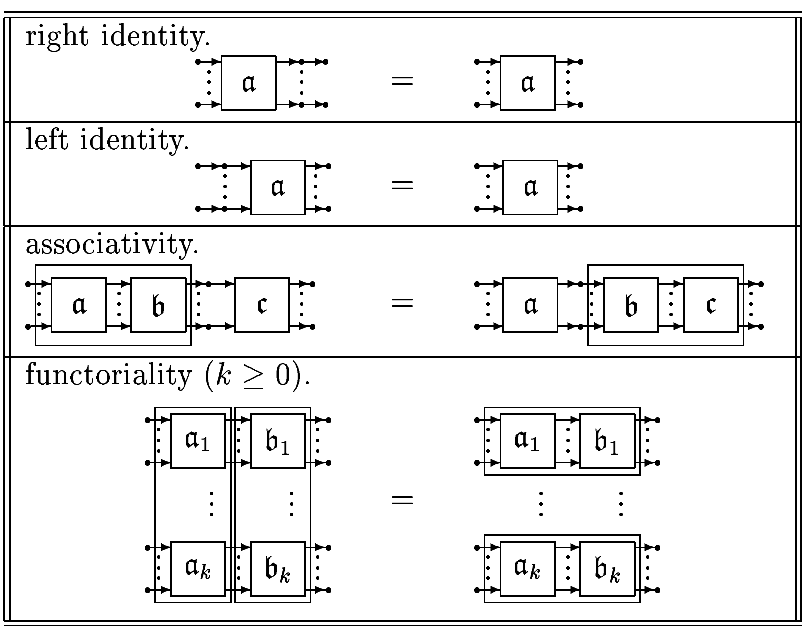


Table 2: Elementary equations between flow graphs.

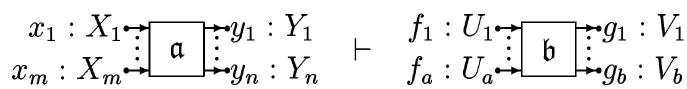
2.17. NOTATION.

- *Occurrences of the form* →•→ *in pictures like those of Tab. 1 and Tab. 2 can be replaced by* ⟶.
- *The sorts* ($X_{\dots}$, $Y_{\dots}$, $Z_{\dots}$) *inside these flow graphs can be omitted if they are not significant.*

2.18. CATEGORY OF FLOW GRAPHS — THE VERTICAL PARTIAL CATEGORY. Due to the equations presented in Tab. 2 flow graphs form a category. The structure of that category factorized by the 'equimorphism' relation which is in account later, will turn out to be 'ps-monoidal' ([CG97]). The graphical representation of morphisms and objects of this category is the same as in [CG97].

2.19. DATA FLOW SPECIFICATIONS. Now, we compose these two categories, the horizontal partial and the vertical partial one, with which we treated to a double category and explain how to describe dynamic aspects of flow graphs.

2.20. DEFINITION. [Data flow rule.] *Given a data signature* Σ *and an agent signature* Ψ *with the same ground sort set* $S := S_\Sigma = S_\Psi$. *An expression*



(22)

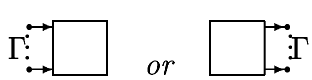
is called a data flow rule *with respect to these signatures with* $\mathfrak{a} \in \mathfrak{hom}_\Psi(\Gamma_1, \Delta_1)$, $\mathfrak{b} \in \mathfrak{hom}_\Psi(\Gamma_2, \Delta_2)$, $m \geq 0$, $n \geq 0$, $a \geq 0$, $b \geq 0$, $(\forall \Gamma_1)\, f_i : U_i \in T_{\Sigma,\Gamma_1,U_i}$ *and* $(\forall \Delta_1)\, g_j : V_j \in T_{\Sigma,\Gamma_2,V_j}$.

2.21. DEFINITION. [Data flow specification.] *A* data flow specification *is a tuple* $\mathbb{Q} := (S_\mathbb{Q}, \Omega_Q, \Xi_\mathbb{Q}, R_\mathbb{Q})$ *where* $\mathbb{Q} := (S_\mathbb{Q}, \Omega_Q)$ *is a data signature and* $\mathbb{Q} := (S_\mathbb{Q}, \Xi_Q)$ *is an agent signature and* $R_\mathbb{Q}$ *is a set of data flow rules* $R_\mathbb{Q}$ *with respect to these signatures.*

2.22. DEFINITION. [Theorem of the data flow logic.] *Given a data flow specification* $\mathbb{Q}$. *The set of* theorems *is the smallest set with respect to set inclusion which contains* $R_\mathbb{Q}$ *and is closed under generation rules of Tab. 3 and Tab. 4. If a data flow rule is a theorem for a data flow specification, then one says, that this rule is* valid *for this specification.*

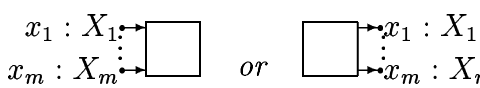
2.23. NOTATION.

- *Let* $\Gamma := \{x_1 : X_1, \dots, x_m : X_m\}$ *be a context.*



resp.

means



resp.

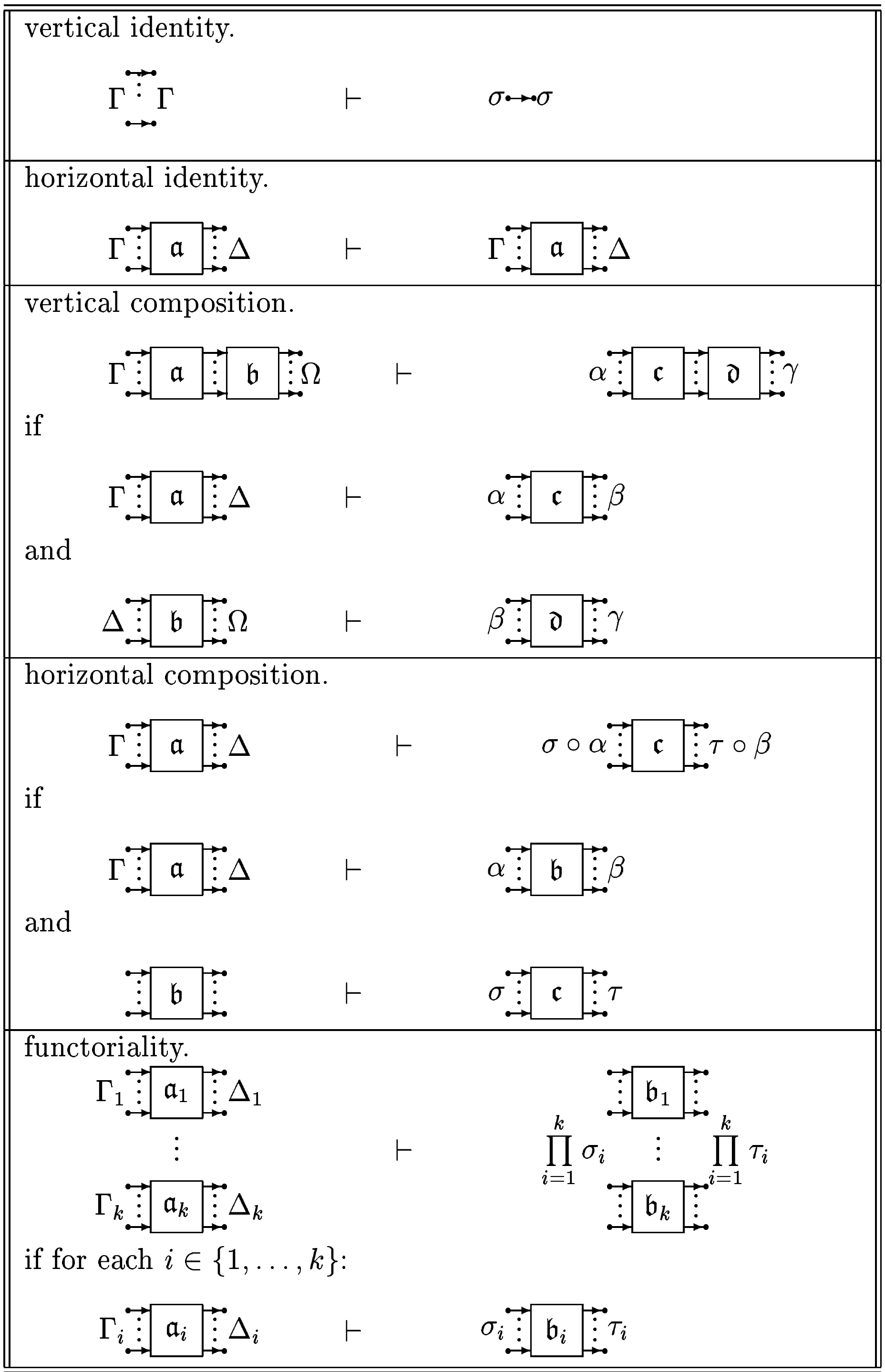


Table 3: Elementary rules on flow graphs — part I.

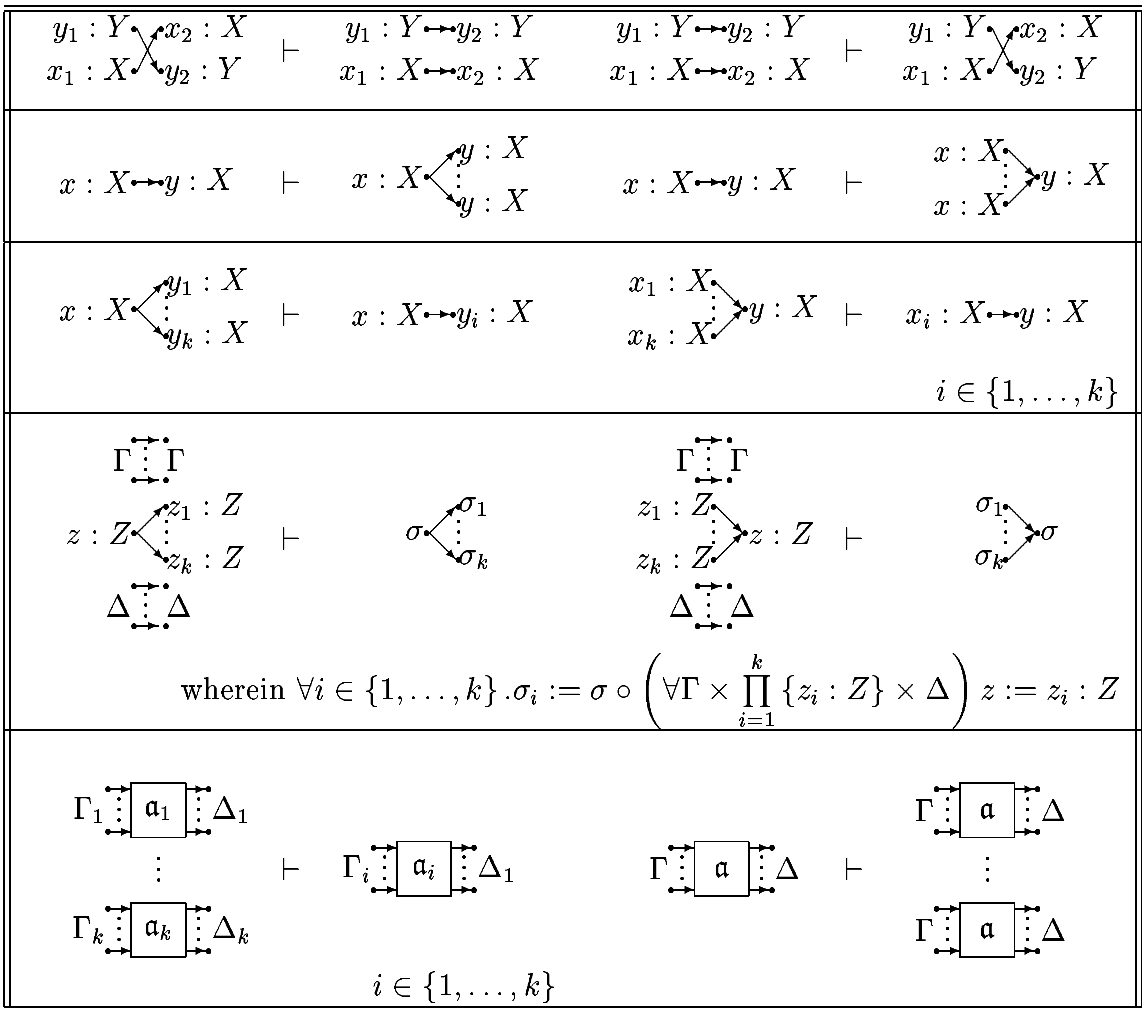
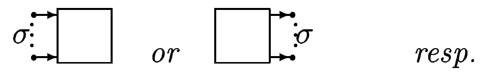
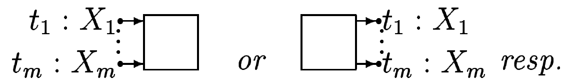


Table 4: Elementary rules on flow graphs — part II.

- *Let* $\sigma := (\forall\Gamma)\, y_1 := t_1 : X_1, \ldots, y_n := t_n : X_n$. *Then*



means



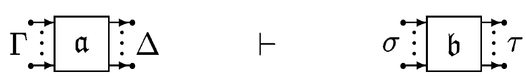
2.24. Equimorphism. The equimorphism concept is the 'equality' of the data flow logic. There are two kinds of equimorphism: the *vertical equimorphism* which can hold between substitutions and the *horizontal equimorphism* which can hold between flow graphs.

2.25. Definition. [Vertical equimorphism.] *Given a data flow specification* $\mathbb{Q}$. *Two terms* $(\forall\Gamma)\, f : Y, (\forall\Gamma)\, g : Y \in T_{\mathbb{Q},\Gamma,Y}$ *are* equimorph, *if following data flow rules are valid in* $\mathbb{Q}$:

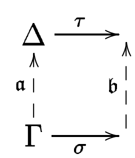
2.26. Definition. [Horizontal equimorphism.] *Given a data flow specification* $\mathbb{Q}$. *Two flow graphs* $\mathfrak{a}, \mathfrak{b} \in \mathfrak{hom}_{\mathbb{Q}}(\Gamma, \Delta)$ *are* equimorph, *if the following data flow rules are valid:*

2.27. Notation. *The left of these relations in these definitions is denoted by* $\sqsubseteq$. *The equimorphism relation is denoted by* $\doteq$.

2.28. Representation of data flow rules by double categories. A data flow rule



relates to the following *square*:



These squares are, as we point out in the next chapter in detail, morphisms of two categories, simultaneously.

Assume, an adder like that in Fig. 1 shall be specified. Fig. 2 shows the necessary data flow rules. The signatures are obviously. And these are the squares relating to the data flow rules of Fig. 2:

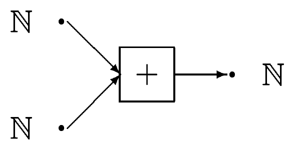


Figure 1: Flow graph for adding two natural numbers.

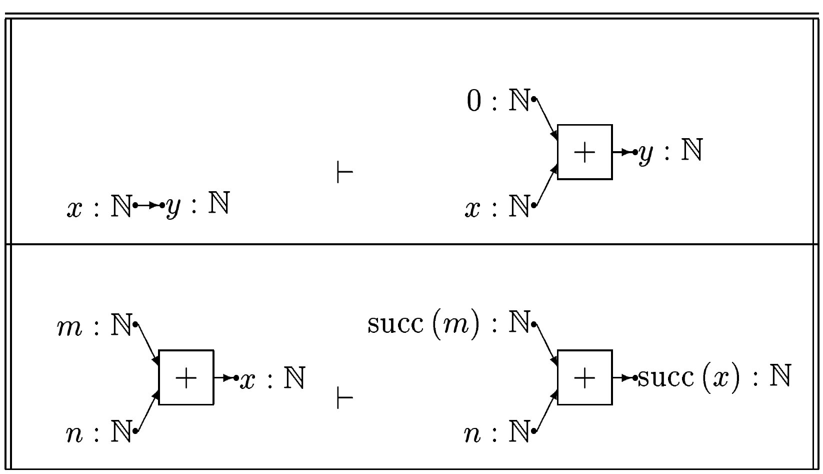


Figure 2: Specification of an adder.

$$\begin{array}{ccc}
\{y:\mathbb{N}\} & \xrightarrow{(\forall\{y:\mathbb{N}\})y:=y:\mathbb{N}} & \{y:\mathbb{N}\} \\
{\scriptstyle \mathfrak{id}_{\mathbb{N}}}\uparrow & & \uparrow{\scriptstyle +} \\
\{x:\mathbb{N}\} & \xrightarrow[(\forall\{x:\mathbb{N}\})m:=0:\mathbb{N},n:=x:\mathbb{N}]{} & \{m:\mathbb{N},n:\mathbb{N}\}
\end{array}$$

$$\begin{array}{ccc}
\{x:\mathbb{N}\} & \xrightarrow{(\forall\{x:\mathbb{N}\})x:=\mathrm{succ}(x):\mathbb{N}} & \{x:\mathbb{N}\} \\
{\scriptstyle +}\uparrow & & \uparrow{\scriptstyle +} \\
\{m:\mathbb{N},n:\mathbb{N}\} & \xrightarrow[(\forall\{m:\mathbb{N},n:\mathbb{N}\})m:=\mathrm{succ}(m):\mathbb{N},n:=n:\mathbb{N}]{} & \{m:\mathbb{N},n:\mathbb{N}\}
\end{array}$$

2.29. Presentation of data flow specifications by equation-less double-categorical constructions. As one can show using the lemmas below data flow specifications can be presented by double categories with finite products and finite products of the double category whose morphisms and squares yield by taking the mirror image at the horizontal axis from the respective morphisms and squares of the former double category.

3. Base terms in the theory of double categories

3.1. Definition. [Double category.] [2] *On the one hand a* double category $\mathbb{C}$ *consists of*

- *a class* $|\mathbb{C}|$ *of* objects.
- *a class* $\mathfrak{mor}_{\mathbb{C}}$ *of* vertical morphisms.
- *a class* $\mathbf{mor}_{\mathbb{C}}$ *of* horizontal morphisms.

where

- *The objects together with the vertical morphisms form a category and the objects together with the horizontal morphisms as well. These categories are called* vertical *and* horizontal partial category*, resp.*

- *The notations as in Tab. 5 (page 15) are used in the sequel.*

On the other hand a double category $\mathbb{C}$ *consists of*

- *a class* $\left(\mathfrak{a} \begin{smallmatrix} b \\ a \end{smallmatrix} \mathfrak{b}\right)_{\mathbb{C}}$ *of* squares *for every four objects* $A, B, C, D \in |\mathbb{C}|$ *and for every four morphisms* $\mathfrak{a} \in \mathfrak{hom}_{\mathbb{C}}(A, C)$, $\mathfrak{b} \in \mathfrak{hom}_{\mathbb{C}}(B, D)$, $a \in \mathbf{hom}_{\mathbb{C}}(A, B)$, $b \in \mathbf{hom}_{\mathbb{C}}(C, D)$,
  whereby

[2]This terminus was introduced in [BE74].

dimension	vertical	horizontal	not significant
set of morphisms between any $A, B \in \lvert\mathbb{C}\rvert$	$\mathfrak{hom}_{\mathbb{C}}(A,B)$	$\mathbf{hom}_{\mathbb{C}}(A,B)$	$\mathrm{Hom}_{\mathbb{C}}(A,B)$
composition	$\bullet$	$\circ$	$\circ$
identity on any $A \in \lvert\mathbb{C}\rvert$	$\mathfrak{id}_A$	$\mathbf{id}_A$	id_A
pictorial presentation of any morphism between $A, B \in \lvert\mathbb{C}\rvert$	$\begin{array}{c} B \\ \mathfrak{h}\uparrow \\ A \end{array}$	$A \xrightarrow{h} B$	

Table 5: Notations in partial categories.

- *their elements Q are depicted as follows:*

$$\begin{array}{ccc} C & \xrightarrow{b} & D \\ \mathfrak{a}\uparrow & Q & \uparrow\mathfrak{b} \\ A & \xrightarrow{a} & B \end{array}$$

- *it is the base for the definition of following classes:*

$$\mathfrak{hom}_{\mathbb{C}}(a,b) := \left\{ Q \in \begin{pmatrix} \mathfrak{a} & \begin{matrix} b \\ a \end{matrix} & \mathfrak{b} \end{pmatrix}_{\mathbb{C}} \middle| \mathfrak{a} \in \mathfrak{hom}_{\mathbb{C}}(A,C), \mathfrak{b} \in \mathfrak{hom}_{\mathbb{C}}(B,D) \right\}$$

$$\mathbf{hom}_{\mathbb{C}}(\mathfrak{a},\mathfrak{b}) := \left\{ Q \in \begin{pmatrix} \mathfrak{a} & \begin{matrix} b \\ a \end{matrix} & \mathfrak{b} \end{pmatrix}_{\mathbb{C}} \middle| a \in \mathbf{hom}_{\mathbb{C}}(A,B), b \in \mathbf{hom}_{\mathbb{C}}(C,D) \right\}$$

there are the following ingredients

- *The square* $\mathfrak{id}_h \in \begin{pmatrix} \mathfrak{id}_A & \begin{matrix} h \\ h \end{matrix} & \mathfrak{id}_B \end{pmatrix}_{\mathbb{C}}$*:*

$$\begin{array}{ccc} A & \xrightarrow{h} & B \\ \mathfrak{id}_A\uparrow & \mathfrak{id}_h & \uparrow\mathfrak{id}_B \\ A & \xrightarrow{h} & B \end{array}$$

  for every two objects $A, B \in \lvert\mathbb{C}\rvert$ and a horizontal morphism $h \in \mathbf{hom}_{\mathbb{C}}(A,B)$

- *The square* $\mathbf{id}_{\mathfrak{h}} \in \begin{pmatrix} \mathfrak{h} & \begin{matrix} \mathbf{id}_B \\ \mathbf{id}_A \end{matrix} & \mathfrak{h} \end{pmatrix}_{\mathbb{C}}$*:*

$$\begin{array}{ccc} B & \xrightarrow{\mathbf{id}_B} & B \\ {\scriptstyle\mathfrak{h}}\uparrow & \mathbf{id}_{\mathfrak{h}} & \uparrow{\scriptstyle\mathfrak{h}} \\ A & \xrightarrow{\mathbf{id}_A} & A \end{array}$$

for every two objects $A, B \in |\mathbb{C}|$ *and vertical morphism* $\mathfrak{h} \in \mathfrak{hom}_{\mathbb{C}}(A, B)$,

- *The* vertical composition $P \bullet Q \in \left(\mathfrak{e} \bullet \mathfrak{b} \ \substack{f \\ b} \ \mathfrak{f} \bullet \mathfrak{c}\right)_{\mathbb{C}}$ *:*

$$\begin{array}{ccc} H & \xrightarrow{f} & I \\ {\scriptstyle\mathfrak{e}}\uparrow & P & \uparrow{\scriptstyle\mathfrak{f}} \\ E & \xrightarrow{k} & F \\ {\scriptstyle\mathfrak{b}}\uparrow & Q & \uparrow{\scriptstyle\mathfrak{c}} \\ B & \xrightarrow{h} & C \end{array}$$

for every appropriate squares, morphisms and objects such it can be performed,

- *The* horizontal composition $P \circ R \in \left(\mathfrak{d} \ \substack{f \circ e \\ d \circ c} \ \mathfrak{f}\right)_{\mathbb{C}}$ *:*

$$\begin{array}{ccccc} G & \xrightarrow{e} & H & \xrightarrow{f} & I \\ {\scriptstyle\mathfrak{d}}\uparrow & R & {\scriptstyle\mathfrak{e}}\uparrow & P & \uparrow{\scriptstyle\mathfrak{f}} \\ D & \xrightarrow{c} & E & \xrightarrow{d} & F \end{array}$$

for every appropriate squares, morphisms and objects such it can be performed,

whereby

- *the horizontal morphisms as the objects, the squares as morphisms and the vertical composition and identity of squares as composition and identity, resp., form a category.*
  In a similar way the vertical morphisms as the objects, the squares as morphisms and the horizontal composition and identity of squares as composition and identity, resp., form a category.

- *The equations below hold:*

  –

$$\mathfrak{id}_{g \circ f} \quad = \quad \mathfrak{id}_g \circ \mathfrak{id}_f \tag{23}$$

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C \\ {\scriptstyle \mathfrak{id}_A}\uparrow & \mathfrak{id}_f & \uparrow{\scriptstyle \mathfrak{id}_B} & \mathfrak{id}_g & \uparrow{\scriptstyle \mathfrak{id}_C} \\ A & \xrightarrow{f} & B & \xrightarrow{g} & C \end{array}$$

for every appropriate objects and horizontal morphisms,

–

$$\mathbf{id}_{\mathfrak{g}\bullet\mathfrak{f}} \quad = \quad \mathbf{id}_{\mathfrak{g}} \bullet \mathbf{id}_{\mathfrak{f}} \tag{24}$$

$$\begin{array}{ccc} C & \xrightarrow{\mathbf{id}_C} & C \\ {\scriptstyle \mathfrak{g}}\uparrow & \mathbf{id}_{\mathfrak{g}} & \uparrow{\scriptstyle \mathfrak{g}} \\ B & \xrightarrow{\mathbf{id}_B} & B \\ {\scriptstyle \mathfrak{f}}\uparrow & \mathbf{id}_{\mathfrak{f}} & \uparrow{\scriptstyle \mathfrak{f}} \\ A & \xrightarrow{\mathbf{id}_A} & A \end{array}$$

for every appropriate objects and vertical morphisms

–

$$id_A \quad := \quad \mathbf{id}_{\mathfrak{id}_A} = \mathfrak{id}_{\mathbf{id}_A} \tag{25}$$

$$\begin{array}{ccc} A & \xrightarrow{\mathbf{id}_A} & A \\ {\scriptstyle \mathfrak{id}_A}\uparrow & id_A & \uparrow{\scriptstyle \mathfrak{id}_A} \\ A & \xrightarrow{\mathbf{id}_A} & A \end{array}$$

for each object $A \in |\mathbb{C}|$,

–

$$(P \circ R) \bullet (Q \circ S) \quad = \quad (P \bullet Q) \circ (R \bullet S) \tag{26}$$

$$\begin{array}{ccccc} G & \xrightarrow{e} & H & \xrightarrow{f} & I \\ {\scriptstyle \mathfrak{d}}\uparrow & R & \uparrow{\scriptstyle \mathfrak{e}} & P & \uparrow{\scriptstyle \mathfrak{f}} \\ D & \xrightarrow{c} & E & \xrightarrow{d} & F \\ {\scriptstyle \mathfrak{a}}\uparrow & S & \uparrow{\scriptstyle \mathfrak{b}} & Q & \uparrow{\scriptstyle \mathfrak{c}} \\ A & \xrightarrow{a} & B & \xrightarrow{b} & C \end{array}$$

for every appropriate objects, morphisms and squares.

3.2. Example. [The double category Pfn of the partial functions.] The sets as objects, the total functions between them as horizontal morphisms and the partial functions as vertical morphisms and classes $\left(\mathfrak{h} \begin{smallmatrix} k \\ h \end{smallmatrix} \mathfrak{k}\right)$ with at most one member

$$\begin{array}{ccc} C & \xrightarrow{k} & D \\ \mathfrak{h}\uparrow & & \uparrow\mathfrak{k} \\ A & \xrightarrow{h} & B \end{array}$$

with

$$\left(\mathfrak{h} \begin{matrix} k \\ h \end{matrix} \mathfrak{k}\right) \neq \emptyset \quad :\text{iff} \quad \forall x \in \mathrm{dom}\mathfrak{h}.h\left(x\right) \in \mathrm{dom}\mathfrak{k} \wedge k\left(\mathfrak{h}\left(x\right)\right) = \mathfrak{k}\left(h\left(x\right)\right) \tag{27}$$

as classes of squares, form the double category Pfn of partial functions.
There is known to be a problem at the treatment of (usual) category Pfn of the partial functions whose objects are the sets and whose morphisms are the partial functions, that the cartesian set product in it is not the product in the sense of usual category theory since the projections of this products are not natural transformations in this category as you can see looking at the following example:

$$\begin{array}{ccccc} \mathbb{R}\times\mathbb{R} & \xrightarrow{\mathrm{rec}\times\mathrm{succ}} & \mathbb{R}\times\mathbb{R} & \xrightarrow{q_{\mathbb{R},\mathbb{R}}} & \mathbb{R} \\ \mathfrak{id}_{\mathbb{R}\times\mathbb{R}}\uparrow & & & & \uparrow\mathfrak{id}_{\mathbb{R}} \\ \mathbb{R}\times\mathbb{R} & \xrightarrow{q_{\mathbb{R},\mathbb{R}}} & \mathbb{R} & \xrightarrow{\mathrm{succ}} & \mathbb{R} \end{array}$$

Here $\mathbb{R}$ denotes the set of rational numbers, rec the function assigning each rational number different from 0 its *reciprocal*, and succ the successor function. the function rec is properly partial with $\mathrm{domrec} = \mathbb{R} \setminus \{0\}$. Now, you can see the above diagram does not commute because

$$\mathrm{dom}q_{\mathbb{R},\mathbb{R}} \circ (\mathrm{rec} \times \mathrm{succ}) = \mathbb{R} \setminus \{0\} \tag{28}$$

$$\mathrm{domsucc} \circ q_{\mathbb{R},\mathbb{R}} = \mathbb{R} \tag{29}$$

However, the condition for the existence of the respective square in the *double category* Pfn is fulfilled.

3.3. Example. [The double category Rel of the binary relations.] The double category Rel of binary relations is very similar to the double category Pfn of the partial functions. The objects and horizontal morphisms of this double category are the same as those of the double category Pfn of the partial functions. The vertical morphisms are the binary relations. Herein it holds for horizontal morphisms:

- The *identity* is the diagonal:

$$\mathbf{id}_A := \{(a, a) \,|a \in A\} \tag{30}$$

  for each $A \in |\mathrm{Rel}|$.

- The *composition* is the composition of relations:

$$R \circ S \;:=\; \{(s, r) \,|\exists x. sSx \wedge xRr\} \tag{31}$$

  for each appropriate binary relations R and S.

- *projections*, *diagonals*, *injections*, and *codiagonals* are the respective partial functions of the double category Pfn viewed as binary relations.

Again, the collection of squares between four respective morphisms has at most one member, where it holds this time:

$$\begin{pmatrix} \mathfrak{h} & \overset{k}{\underset{h}{}} & \mathfrak{k} \end{pmatrix} \neq \emptyset \quad \text{: iff} \quad \forall x \in A. y \in C. xhy \to (h(x))\, \mathfrak{k}\, (k(y)) \tag{32}$$

3.4. Example. [The double category Pos of partial orders.] The objects of this double category are the *partial ordered sets* $M = (M, \leq_M)$ with M a set and $\leq_M \subseteq M \times M$ a partial order relation, i. e. a binary relation on M satisfying the following properties:

Reflexivity.

$$\frac{}{x \leq_M x} \tag{33}$$

Transitivity.

$$\frac{x \leq_M y \quad y \leq_M z}{x \leq_M z} \tag{34}$$

Antisymmetry.

$$\frac{x \leq_M y \quad y \leq_M x}{x = y} \tag{35}$$

for each $x, y, z \in M$.

The vertical morphisms of this category between two partial ordered sets M and N are *monotonous* relations, i. e. relations $R \subseteq M \times N$ with

$$\frac{u \leq_M v \quad vRx \quad x \leq_N y}{uRy} \quad \text{for each } u, v \in M,\ x, y \in N. \tag{36}$$

The vertical identity on a partial ordered set M is the partial order relation $\leq_M$.

The horizontal morphisms between the same objects are the *monotonous functions* , i. e. total functions $f \in M \to N$ with

$$\frac{x \leq_M y}{f(x) \leq_N f(y)} \quad \text{for each } x, y \in M. \tag{37}$$

In a similar fashion the squares are the squares of the double category Rel (see example 3.3).

3.5. REMARK. [Schlicht double categories.] [3] In the double categories Pfn of the partial functions, Rel of the binary relations and Pos of the partial orders from the preceding examples there is at most one square for every two vertical morphisms $\mathfrak{h}$ and $\mathfrak{k}$ and horizontal morphisms h and k of appropriate types. Also in the sequel we only use such double categories. They are called *schlicht*. If $\left(\mathfrak{h} \begin{smallmatrix} k \\ h \end{smallmatrix} \mathfrak{k}\right) \neq \emptyset$, we will say the square between the above-mentioned morphisms *exists* or *holds*. Very often and in the following cases the schlicht-ness condition can be forgotten by supposing that squares mean 'proofs'. Two squares between the same morphisms denote two proofs proving the same thing.

3.6. DEFINITION. [Functor.] *Given two double categories $\mathbb{C}$ and $\mathbb{D}$. A map F from objects, vertical and horizontal morphisms and squares of the double category $\mathbb{C}$ to the respective parts of the double category $\mathbb{D}$ is called a* functor *between the double categories $\mathbb{C}$ and $\mathbb{D}$ if it is a functor in the usual sense of category theory between all respective categories forming the double categories $\mathbb{C}$ and $\mathbb{D}$, resp., i. e. categories of the horizontal and vertical morphisms and the categories of the squares with their respectively horizontal or vertical composition.*

3.7. DEFINITION. [Paramorphism.] *Given two double categories $\mathbb{C}$ and $\mathbb{D}$ and two functors F and G between them. A tuple of families $\mathfrak{n} = (\mathfrak{n}, \mathfrak{n})$ with*

$$\mathfrak{n} = (\mathfrak{n}_X \in \mathfrak{hom}_{\mathbb{D}}(F(X), G(X)) \,|X \in |\mathbb{C}|) \tag{38}$$

$$\mathfrak{n} = (\mathfrak{n}_h \in \mathfrak{hom}_{\mathbb{D}}(F(h), G(h)) \,|h \in \mathbf{mor}_{\mathbb{C}}) \tag{39}$$

is called vertical paramorphism *('parametric morphism') and a tuple of families $n = (n, n)$ with*

$$n = (n_X \in \mathbf{hom}_{\mathbb{D}}(F(X), G(X)) \,|X \in |\mathbb{C}|) \tag{40}$$

$$n = (n_{\mathfrak{h}} \in \mathbf{hom}_{\mathbb{D}}(F(\mathfrak{h}), G(\mathfrak{h})) \,|\mathfrak{h} \in \mathfrak{mor}_{\mathbb{C}}) \tag{41}$$

is called horizontal paramorphism*, iff for every appropriate objects and morphisms there is a square $\mathfrak{n}_h \in \left(\mathfrak{n}_A \begin{smallmatrix} G(h) \\ F(h) \end{smallmatrix} \mathfrak{n}_B\right)_{\mathbb{C}}$ or $n_{\mathfrak{h}} \in \left(F(\mathfrak{h}) \begin{smallmatrix} n_B \\ n_A \end{smallmatrix} G(\mathfrak{h})\right)_{\mathbb{C}}$, resp.:*

$$\begin{array}{ccc} G(A) & \xrightarrow{G(h)} & G(B) \\ \uparrow \mathfrak{n}_A & \mathfrak{n}_h & \uparrow \mathfrak{n}_B \\ F(A) & \xrightarrow{F(h)} & F(B) \end{array} \qquad \textit{or} \qquad \begin{array}{ccc} F(B) & \xrightarrow{n_B} & G(B) \\ \uparrow F(\mathfrak{h}) & n_{\mathfrak{h}} & \uparrow G(\mathfrak{h}) \\ F(A) & \xrightarrow{n_A} & G(A) \end{array} \qquad \textit{resp.}$$

and for every appropriate objects and morphisms it holds: $\mathfrak{n}_{a \circ b} = \mathfrak{n}_a \circ \mathfrak{n}_b$ or $n_{\mathfrak{a} \bullet \mathfrak{b}} = n_{\mathfrak{a}} \bullet n_{\mathfrak{b}}$, resp..

[3]'Schlicht' is a German word — it means 'artless' or 'simple', sometimes also 'plain' or 'sober'.

$$\begin{array}{ccccc} G(A) & \xrightarrow{G(b)} & G(B) & \xrightarrow{G(a)} & G(C) \\ \mathfrak{n}_A \uparrow & \mathfrak{n}_b & \mathfrak{n}_B \uparrow & \mathfrak{n}_a & \mathfrak{n}_C \uparrow \\ F(A) & \xrightarrow{F(b)} & F(B) & \xrightarrow{F(a)} & F(C) \end{array} \quad or \quad \begin{array}{ccc} F(C) & \xrightarrow{n_C} & G(C) \\ F(\mathfrak{b}) \uparrow & n_{\mathfrak{b}} & G(\mathfrak{b}) \uparrow \\ F(B) & \xrightarrow{n_B} & G(B) \\ F(\mathfrak{a}) \uparrow & n_{\mathfrak{a}} & G(\mathfrak{a}) \uparrow \\ F(A) & \xrightarrow{n_A} & G(A) \end{array} \quad resp.$$

Let $\mathbb{C}$ and $\mathbb{D}$ be two double categories. The functors as the objects and the paramorphisms in the respective dimension between them as morphisms form a double category $\mathbb{C} \Rightarrow \mathbb{D}$. In this double category there is a square Q between appropriate paramorphisms between appropriate functors

$$\begin{array}{ccc} G & \xrightarrow{b} & B \\ \mathfrak{m} \uparrow & Q & \mathfrak{n} \uparrow \\ F & \xrightarrow{a} & A \end{array}$$

iff there is a family of squares $(Q_X | X \in |\mathbb{C}|)$ with

$$\begin{array}{ccc} G(X) & \xrightarrow{b_X} & B(X) \\ \mathfrak{m}_X \uparrow & Q_X & \mathfrak{n}_X \uparrow \\ F(X) & \xrightarrow{a_X} & A(X) \end{array}$$

3.8. Notation. *Let $\mathbb{C}$, $\mathbb{D}$ and $\mathbb{E}$ be three double categories, $F, G \in |\mathbb{C} \Rightarrow \mathbb{D}|$, $A, B \in |\mathbb{D} \Rightarrow \mathbb{E}|$ functors between the respective double categories and two paramorphisms: $n \in Hom_{\mathbb{C}\Rightarrow\mathbb{D}}(F, G)$ and $k \in Hom_{\mathbb{D}\Rightarrow\mathbb{E}}(A, B)$. The following expressions denote some paramorphisms:*

$$n \cdot A := \left(n_{A(X)} | X\right) \tag{42}$$
$$F \cdot k := \left(F(k_X) | X\right) \tag{43}$$

with the X as respective class of pieces of a double category.

3.9. Remark. There is the following square as you can readily see:

$$\begin{array}{ccc} F \circ B & \xrightarrow{n \cdot B} & G \circ B \\ F \cdot k \uparrow & & G \cdot k \uparrow \\ F \circ A & \xrightarrow{n \cdot A} & G \circ A \end{array} \qquad \begin{array}{ccc} G \circ A & \xrightarrow{G \cdot k} & G \circ B \\ n \cdot A \uparrow & & n \cdot B \uparrow \\ F \circ A & \xrightarrow{F \cdot k} & F \circ B \end{array}$$

3.10. REMARK. The $\_ \cdot \_$-operation is associative:

$$
\begin{aligned}
m \cdot B \cdot V &:= (m \cdot B) \cdot V = m \cdot (B \circ V) && (44)\\
U \cdot m \cdot B &:= (U \cdot m) \cdot B = U \cdot (m \cdot B) && (45)\\
R \cdot U \cdot m &:= (R \circ U) \cdot m = R \cdot (U \cdot m) && (46)
\end{aligned}
$$

for respective paramorphisms m and functors R, B, U, V, but it must be *warned* you not to regard $\_\cdot\_$ as an bifunctor. $\_\cdot\_$ is *not* a bifunctor, i. e. $(m \cdot B)\circ(F \cdot n) = (G \cdot n)\circ(m \cdot A)$ does *not* hold with respective functors A, B, F, G and paramorphisms m, n. That means, the (small) double categories, the functors between them and the paramorphisms of both dimensions do *not* form a 2-category!

4. Equation-less constructions in double categories

4.1. DEFINITION. [Equimorphism.] *Given a double category* $\mathbb{C}$. *An object or a horizontal morphism* h *is called* vertically equimorph *with respect to an object or a horizontal morphism* k, *resp., iff*

$$
\begin{aligned}
\begin{pmatrix} & k & \\ \mathfrak{id}_A & & \mathfrak{id}_B \\ & h & \end{pmatrix}_{\mathbb{C}} &\neq \emptyset && (47)\\
\begin{pmatrix} & h & \\ \mathfrak{id}_A & & \mathfrak{id}_B \\ & k & \end{pmatrix}_{\mathbb{C}} &\neq \emptyset && (48)
\end{aligned}
$$

or graphically

$$
\begin{array}{ccc}
A & \xrightarrow{h} & B \\
\mathfrak{id}_A \Big\Uparrow & & \Big\Uparrow \mathfrak{id}_B \\
A & \xrightarrow{k} & B
\end{array}
\qquad
\begin{array}{ccc}
A & \xrightarrow{k} & B \\
\mathfrak{id}_A \Big\Uparrow & & \Big\Uparrow \mathfrak{id}_B \\
A & \xrightarrow{h} & B
\end{array}
$$

Accordingly the terminus of the horizontal equimorphism *is dual to it.*

4.2. NOTATION. *We also write, if the first square holds,* $k \sqsubseteq h$, *and if both squares hold, i. e.* h *and* k *are equimorph,* $h \doteq k$.

4.3. REMARK. One can easily see, that the vertical (horizontal) $\sqsubseteq$-relation is a pre-order relation (reflexive; transitive) with respect to which the horizontal (vertical) composition and the application of functors is monotonous.

4.4. EXAMPLE. [The double category Pfn of the partial functions.] (see example 3.2; page 18) For two partial functions, i. e. vertical morphisms $\mathfrak{h}$ and $\mathfrak{k}$ in a double category Pfn between the same sets $\mathfrak{h} \sqsubseteq \mathfrak{k}$ holds iff $\mathrm{dom}\mathfrak{h} \subseteq \mathrm{dom}\mathfrak{k}$ and $\mathfrak{h} = \mathfrak{k} \downarrow \mathrm{dom}\mathfrak{h}$.
For two horizontal morphisms h and k, i. e. total functions, between the same sets $h \sqsubseteq k$ holds iff $h = k$. A characteristic property of this relation $\sqsubseteq$ is its symmetry.

4.5. Example. [The double category Rel of the binary relations.] (see example 3.3; page 18) For two binary relations, i. e. vertical morphisms $\mathfrak{h}$ and $\mathfrak{k}$ in the double category Rel between the same sets $\mathfrak{h} \sqsubseteq \mathfrak{k}$ holds iff for this relation interpreted as a set $\mathfrak{h} \subseteq \mathfrak{k}$ holds. Between two total functions between the same sets, i. e. horizontal morphisms h and k in the double category Rel $h \sqsubseteq k$ holds iff $h = k$ holds. The symmetry property of the relation $\sqsubseteq$ follows from the fact, that there is a relation $R^o \subseteq Y \times X$:

$$yR^o x \quad :\text{iff} \quad xRy \quad \text{for each } x \in X, y \in Y. \tag{49}$$

for each binary relation $R \subseteq X \times Y$ where the following conditions are fulfilled:

1. $$\mathfrak{id}_X^o \doteq \mathfrak{id}_X \tag{50}$$

2. For each existing square

$$\begin{array}{ccc} Y & \xrightarrow{k} & A \\ {\scriptstyle R}\uparrow & & \uparrow{\scriptstyle S} \\ X & \xrightarrow{h} & B \end{array}$$

there is a square

$$\begin{array}{ccc} X & \xrightarrow{h} & B \\ {\scriptstyle R^o}\uparrow & & \uparrow{\scriptstyle S^o} \\ Y & \xrightarrow{k} & A \end{array}$$

Therefore if $h \sqsubseteq k$ holds, so $k \sqsubseteq h$ holds as well.

4.6. Example. [The double category Pos of partial orders.] (cf. example 3.4, page 19). Between two vertical morphisms, i. e. monotonous relations R and S between the same objects the $\sqsubseteq$ relation holds iff $R \subseteq S$. Between horizontal morphisms, i. e. monotonous functions f and g between the same objects the relation $\sqsubseteq$ holds iff $f(x) \leq g(x)$ is fulfilled for each appropriate x.

4.7. Definition. [Isomorphism.] *Given a double category $\mathbb{C}$. Two objects A and B are called* isomorph *to each other, noted: $A \cong B$, iff there is an* isomorphism *between A and B. These are two vertical morphisms $\mathfrak{h} \in \mathfrak{hom}_{\mathbb{C}}(A, B)$, $\mathfrak{h}_* \in \mathfrak{hom}_{\mathbb{C}}(B, A)$ and two horizontal morphisms $h \in \mathbf{hom}_{\mathbb{C}}(A, B)$, $h_* \in \mathbf{hom}_{\mathbb{C}}(B, A)$, such that the following squares exists:*

$$\begin{array}{ccc} A & \xrightarrow{h} & B \\ {\scriptstyle \mathfrak{id}_A}\uparrow & & \uparrow{\scriptstyle \mathfrak{h}} \\ A & \xrightarrow{\mathbf{id}_A} & A \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{h} & B \\ {\scriptstyle \mathfrak{h}_*}\uparrow & & \uparrow{\scriptstyle \mathfrak{id}_B} \\ B & \xrightarrow{\mathbf{id}_B} & B \end{array} \qquad \begin{array}{ccc} B & \xrightarrow{h_*} & A \\ {\scriptstyle \mathfrak{id}_B}\uparrow & & \uparrow{\scriptstyle \mathfrak{h}_*} \\ B & \xrightarrow{\mathbf{id}_B} & B \end{array} \qquad \begin{array}{ccc} B & \xrightarrow{h_*} & A \\ {\scriptstyle \mathfrak{h}}\uparrow & & \uparrow{\scriptstyle \mathfrak{id}_A} \\ A & \xrightarrow{\mathbf{id}_A} & A \end{array}$$

$$\begin{array}{ccc} A & \xrightarrow{\mathbf{id}_A} & A \\ {\scriptstyle \mathfrak{id}_A}\uparrow & & \uparrow{\scriptstyle \mathfrak{h}_*} \\ A & \xrightarrow{h} & B \end{array} \qquad \begin{array}{ccc} B & \xrightarrow{\mathbf{id}_B} & B \\ {\scriptstyle \mathfrak{h}}\uparrow & & \uparrow{\scriptstyle \mathfrak{id}_B} \\ A & \xrightarrow{h} & B \end{array} \qquad \begin{array}{ccc} B & \xrightarrow{\mathbf{id}_B} & B \\ {\scriptstyle \mathfrak{id}_B}\uparrow & & \uparrow{\scriptstyle \mathfrak{h}} \\ B & \xrightarrow{h_*} & A \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{\mathbf{id}_A} & A \\ {\scriptstyle \mathfrak{h}_*}\uparrow & & \uparrow{\scriptstyle \mathfrak{id}_A} \\ B & \xrightarrow{h_*} & A \end{array}$$

Two functors are isomorph iff they are isomorph objects in the respective functor category.

4.8. Remark. Obviously, the isomorphism relation is an equivalence relation like the equimorphism relation.

4.9. Lemma.

- *Let $h \,/\, h_*$ be an isomorphism between two objects A and B in a double category $\mathbb{C}$. In both dimensions it holds:*

$$h \circ h_* \doteq id_A \tag{51}$$
$$h_* \circ h \doteq id_B \tag{52}$$

- *Given two isomorph functors F and G between double categories $\mathbb{C}$ and $\mathbb{D}$. Let $h \,/\, h_*$ be an isomorphism between these functors. Then h and h_* are natural isomorphisms between F and G on the partial categories factorized by the equimorphism relation in both dimensions (vertical and horizontal, resp.), i. e. for each $x \in Hom_{\mathbb{C}}(X, Y)$ in both dimensions holds: $h_Y \circ F(x) \doteq G(x) \circ h_X$, $h_{*,X} \circ G(x) \doteq F(x) \circ h_X$ and $h \circ h_* \doteq id_G$, $h_* \circ h \doteq id_F$.*

Proof. We show this exemplarily. The other parts of the proof are dual to it.

$$\begin{array}{ccccccc} F(X) & \xrightarrow{\mathbf{id}_{F(X)}} & F(X) & \xrightarrow{F(x)} & F(Y) & \xrightarrow{h_Y} & G(Y) \\ {\scriptstyle \mathfrak{id}_{F(X)}}\uparrow & & \uparrow{\scriptstyle h_{*,X}} & & \uparrow{\scriptstyle h_{*,Y}} & & \uparrow{\scriptstyle \mathfrak{id}_{G(Y)}} \\ F(X) & \xrightarrow{h_X} & G(X) & \xrightarrow{G(x)} & G(Y) & \xrightarrow{\mathbf{id}_{G(Y)}} & G(Y) \end{array}$$

$$\begin{array}{ccccccc} F(X) & \xrightarrow{h_{*,X}} & G(X) & \xrightarrow{G(x)} & G(Y) & \xrightarrow{\mathbf{id}_{G(Y)}} & G(Y) \\ {\scriptstyle \mathfrak{id}_{F(X)}}\uparrow & & \uparrow{\scriptstyle h_X} & & \uparrow{\scriptstyle h_Y} & & \uparrow{\scriptstyle \mathfrak{id}_{G(Y)}} \\ F(X) & \xrightarrow{\mathbf{id}_{F(X)}} & F(X) & \xrightarrow{F(x)} & F(Y) & \xrightarrow{h_{*,Y}} & G(Y) \end{array}$$

$$\begin{array}{ccccc} G & \xrightarrow{h_*} & F & \xrightarrow{h} & G \\ {\scriptstyle \mathfrak{id}_G}\uparrow & & \uparrow{\scriptstyle \mathfrak{h}_*} & & \uparrow{\scriptstyle \mathfrak{id}_G} \\ G & \xrightarrow{\mathbf{id}_G} & G & \xrightarrow{\mathbf{id}_G} & G \end{array} \qquad \begin{array}{ccccc} F & \xrightarrow{\mathbf{id}_F} & F & \xrightarrow{\mathbf{id}_F} & F \\ {\scriptstyle \mathfrak{id}_F}\uparrow & & \uparrow{\scriptstyle \mathfrak{h}} & & \uparrow{\scriptstyle \mathfrak{id}_F} \\ F & \xrightarrow{h} & G & \xrightarrow{h_*} & F \end{array}$$

∎

4.10. Definition. [Equation-less adjunction.] *Given two double categories $\mathbb{C}$ and $\mathbb{D}$. A tuple (F, U, η, ϵ) with $F \in |\mathbb{C} \Rightarrow \mathbb{D}|$, $U \in |\mathbb{D} \Rightarrow \mathbb{C}|$, $\eta \in Hom_{\mathbb{C} \Rightarrow \mathbb{C}}(Id_{\mathbb{C}}, UF)$, $\epsilon \in Hom_{\mathbb{D} \Rightarrow \mathbb{D}}(FU, Id_{\mathbb{D}})$ two paramorphisms of both directions is called an* adjunction *between $\mathbb{C}$ and $\mathbb{D}$, if there are the following squares:*

$$
\begin{array}{ccc}
FUF & \xrightarrow{\epsilon \cdot F} & F \\
{\scriptstyle F\cdot\eta}\uparrow & & \uparrow{\scriptstyle \mathfrak{id}_F} \\
F & \xrightarrow{\mathbf{id}_F} & F
\end{array}
\qquad
\begin{array}{ccc}
UFU & \xrightarrow{U \cdot \epsilon} & U \\
{\scriptstyle \eta\cdot U}\uparrow & & \uparrow{\scriptstyle \mathfrak{id}_U} \\
U & \xrightarrow{\mathbf{id}_U} & U
\end{array}
\qquad
\begin{array}{ccc}
Id_{\mathbb{C}} & \xrightarrow{\eta} & UF \\
{\scriptstyle \mathfrak{id}_{Id_{\mathbb{C}}}}\uparrow & & \uparrow{\scriptstyle \eta} \\
Id_{\mathbb{C}} & \xrightarrow{\mathbf{id}_{Id_{\mathbb{C}}}} & Id_{\mathbb{C}}
\end{array}
$$

$$
\begin{array}{ccc}
F & \xrightarrow{\mathbf{id}_F} & F \\
{\scriptstyle \mathfrak{id}_F}\uparrow & & \uparrow{\scriptstyle \epsilon\cdot F} \\
F & \xrightarrow{F \cdot \eta} & FUF
\end{array}
\qquad
\begin{array}{ccc}
U & \xrightarrow{\mathbf{id}_U} & U \\
{\scriptstyle \mathfrak{id}_U}\uparrow & & \uparrow{\scriptstyle U\cdot\epsilon} \\
U & \xrightarrow{\eta \cdot U} & UFU
\end{array}
\qquad
\begin{array}{ccc}
Id_{\mathbb{D}} & \xrightarrow{\mathbf{id}_{Id_{\mathbb{D}}}} & Id_{\mathbb{D}} \\
{\scriptstyle \epsilon}\uparrow & & \uparrow{\scriptstyle \mathfrak{id}_{Id_{\mathbb{D}}}} \\
FU & \xrightarrow{\epsilon} & Id_{\mathbb{D}}
\end{array}
$$

the functor F is called left adjoint*, the functor U is called* right adjoint*, the paramorphism η* unit *and the paramorphism ϵ* counit *of this adjunction.*

4.11. Equation-less finite (co-)products. We will call the functor U in an adjunction (F, U, η, ϵ) with $F := (\_)_n$ between the double categories $\mathbb{C}$ and $(\mathbb{C})_n$ a *n-product-functor*.
We will call the functor F in an adjunction (F, U, η, ϵ) with $U := (\_)_n$ between the double categories $\mathbb{C}$ and $(\mathbb{C})_n$ a *n-coproduct-functor*.
Here $(\mathbb{C})_n$ denotes the double category of the n-tuple of pieces (objects / morphisms of both dimensions / squares) of the double category $\mathbb{C}$:

$$
\begin{aligned}
|(\mathbb{C})_n| &= \left\{\vec{A} | \forall i \in \{1, \ldots, n\} . A_i \in |\mathbb{C}|\right\} && (53)\\
\mathfrak{hom}_{(\mathbb{C})_n}\left(\vec{A}, \vec{B}\right) &= \left\{\vec{\mathfrak{h}} | \forall i \in \{1, \ldots, n\} . \mathfrak{h}_i \in \mathfrak{hom}_{\mathbb{C}}(A_i, B_i)\right\} && (54)\\
\mathbf{hom}_{(\mathbb{C})_n}\left(\vec{A}, \vec{B}\right) &= \left\{\vec{h} | \forall i \in \{1, \ldots, n\} . h_i \in \mathbf{hom}_{\mathbb{C}}(A_i, B_i)\right\} && (55)
\end{aligned}
$$

where $\vec{A} := (A_1, \ldots, A_n)$, $\vec{B} := (B_1, \ldots, B_n)$, $\vec{\mathfrak{h}} := (\mathfrak{h}_1, \ldots, \mathfrak{h}_n)$ and $\vec{h} := (h_1, \ldots, h_n)$ and similarly for the squares and $(\_)_n$ denotes the following functor with X an arbitrary piece of the double category $\mathbb{C}$:

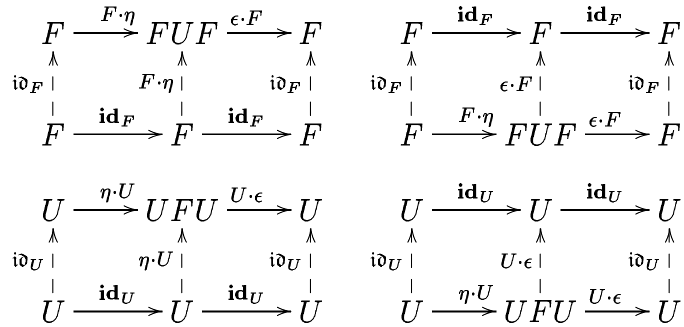
$$
(\_)_n : X \mapsto (X, \ldots, X) \qquad (56)
$$

. The objects of the images of the n-(co-)product-functors for each n-tuple $(A_1, \ldots, A_n)$ with $A_i \in |\mathbb{C}|$ for each $i \in \{1, \ldots, n\}$ is called the n-fold *product / coproduct* of the A_i.

4.12. Lemma. *Given two double categories $\mathbb{C}$ and $\mathbb{D}$ and an adjunction (F, U, η, ϵ) between them, then*

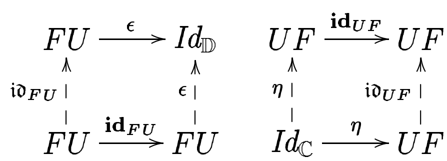
$$
\begin{aligned}
(U \cdot \epsilon) \circ (\eta \cdot U) &\doteq id_U && (57)\\
(\epsilon \cdot F) \circ (F \cdot \eta) &\doteq id_F && (58)
\end{aligned}
$$

Proof.

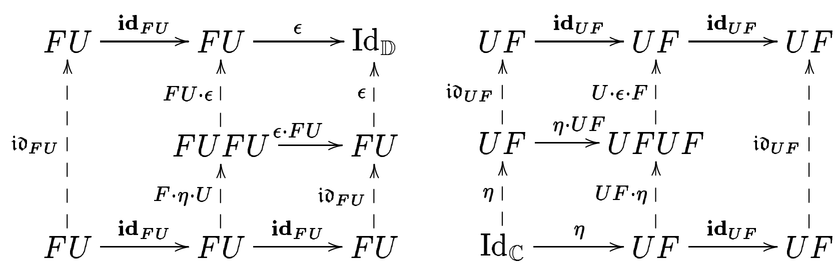


■

4.13. Lemma. *Given an adjunction (F,U,η,ϵ) between the double categories $\mathbb{C}$ and $\mathbb{D}$. You can show the existence of the following squares:*



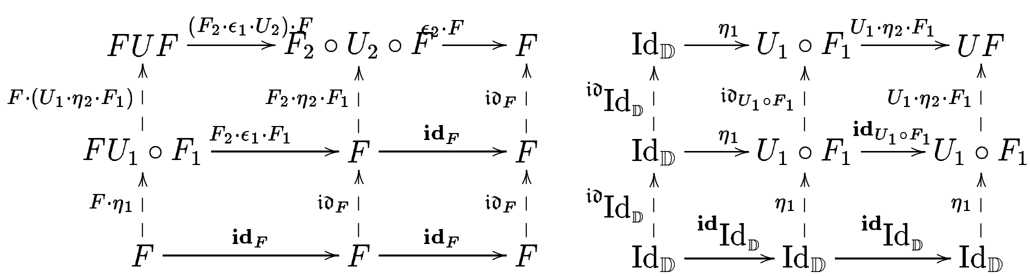
Proof.



and lemma 4.12. ■

4.14. Lemma. *Given three double categories $\mathbb{C}$, $\mathbb{D}$ and $\mathbb{E}$, an adjunction $(F_1,U_1,\eta_1,\epsilon_1)$ between $\mathbb{C}$ and $\mathbb{D}$ and an adjunction $(F_2,U_2,\eta_2,\epsilon_2)$ between $\mathbb{D}$ and $\mathbb{E}$. Then (F,U,η,ϵ) where $F := F_2 \circ F_1$, $U := U_1 \circ U_2$, $\eta := (U_1 \cdot \eta_2 \cdot F_1) \circ \eta_1$, $\epsilon := \epsilon_2 \circ (F_2 \cdot \epsilon_1 \cdot U_2)$ is an adjunction between $\mathbb{C}$ and $\mathbb{E}$.*

Proof. Simple: Systematical computing of the diagrams of the definition of the terminus 'adjunction', e. g.



and similarly for the other squares. ■

4.15. NOTATION. *Given an adjunction* (F, U, η, ϵ).

$$\langle h \rangle := U(h) \circ \eta_X \quad h \in Hom_{\mathbb{D}}(F(X), Y) \tag{59}$$
$$[k] := \epsilon_Y \circ F(k) \quad k \in Hom_{\mathbb{C}}(X, U(Y)) \tag{60}$$
$$x^* := (x \cdot F) \circ (H \cdot \eta) \quad x \in Hom_{\mathbb{D} \Rightarrow \mathbb{E}}(H \circ U, Z) \tag{61}$$
$$y^{\otimes} := (Z \cdot \epsilon) \circ (y \cdot U) \quad y \in Hom_{\mathbb{C} \Rightarrow \mathbb{E}}(H, Z \circ F) \tag{62}$$

4.16. REMARK. In principle, a statement of the following lemma is, that these maps form GALOIS connections:

4.17. LEMMA. *Given two double categories* $\mathbb{C}$ *and* $\mathbb{D}$ *and an adjunction* (F, U, η, ϵ). *In all the dimensions it holds:*

1.

$$[\langle g \rangle] \sqsubseteq g \quad g \in Hom_{\mathbb{D}}(F(X), Y) \tag{63}$$
$$f \sqsubseteq \langle [f] \rangle \quad f \in Hom_{\mathbb{C}}(X, U(Y)) \tag{64}$$

2.

$$f \sqsubseteq \langle g \rangle \quad \textit{iff} \quad [f] \sqsubseteq g \qquad f \in Hom_{\mathbb{C}}(X, U(Y)), \; g \in Hom_{\mathbb{D}}(F(X), Y) \tag{65}$$

3.

$$\langle g \rangle \doteq \langle [\langle g \rangle] \rangle \quad g \in Hom_{\mathbb{D}}(F(X), Y) \tag{66}$$
$$[f] \doteq [\langle [f] \rangle] \quad f \in Hom_{\mathbb{C}}(X, U(Y)) \tag{67}$$

4.

$$U(h) \doteq \langle h \circ \epsilon_X \rangle \quad h \in Hom_{\mathbb{C}}(X, Y) \tag{68}$$
$$F(k) \doteq [\eta_Y \circ k] \quad k \in Hom_{\mathbb{D}}(X, Y) \tag{69}$$

5.

$$\epsilon_Y \circ FU(h) \sqsubseteq h \circ \epsilon_X \quad h \in Hom_{\mathbb{C}}(X, Y) \tag{70}$$
$$\eta_Y \circ k \sqsubseteq UF(k) \circ \eta_X \quad k \in Hom_{\mathbb{D}}(X, Y) \tag{71}$$

6.

$$\langle [U(h)] \rangle \doteq U(h) \quad h \in Hom_{\mathbb{C}}(X,Y) \tag{72}$$
$$[\langle F(k) \rangle] \doteq F(k) \quad k \in Hom_{\mathbb{D}}(X,Y) \tag{73}$$

or in reduced form with the same notations:

$$(U \cdot \epsilon)_Y \circ UFU(h) \circ (\eta \cdot U)_X \doteq U(h) \tag{74}$$
$$(\epsilon \cdot F)_Y \circ FUF(k) \circ (F \cdot \eta)_X \doteq F(k) \tag{75}$$

7.

$$(x^*)^{\otimes} \sqsubseteq x \quad x \in Hom_{\mathbb{D} \Rightarrow \mathbb{E}}(H \circ U, Z) \tag{76}$$
$$y \sqsubseteq \left(y^{\otimes}\right)^* \quad y \in Hom_{\mathbb{C} \Rightarrow \mathbb{E}}(H, Z \circ F) \tag{77}$$

8.

$$y^{\otimes} \sqsubseteq x \quad \textit{iff} \quad y \sqsubseteq x^* \qquad x \in Hom_{\mathbb{D} \Rightarrow \mathbb{E}}(H \circ U, Z) \qquad y \in Hom_{\mathbb{C} \Rightarrow \mathbb{E}}(H, Z \circ F) \tag{78}$$

9.

$$\left((x^*)^{\otimes}\right)^* \doteq x^* \quad x \in Hom_{\mathbb{D} \Rightarrow \mathbb{E}}(H \circ U, Z) \tag{79}$$
$$\left((y^{\otimes})^*\right)^{\otimes} \doteq y^{\otimes} \quad y \in Hom_{\mathbb{C} \Rightarrow \mathbb{E}}(H, Z \circ F) \tag{80}$$

10.

$$h \cdot F \doteq (h \circ (G \cdot \epsilon))^* \quad h \in Hom_{\mathbb{D} \Rightarrow \mathbb{E}}(G, H) \tag{81}$$
$$k \cdot U \doteq ((H \cdot \eta) \circ k)^{\otimes} \quad k \in Hom_{\mathbb{C} \Rightarrow \mathbb{E}}(G, H) \tag{82}$$

11.

$$(H \cdot \epsilon) \circ (h \cdot FU) \sqsubseteq h \circ (G \cdot \epsilon) \quad h \in Hom_{\mathbb{D} \Rightarrow \mathbb{E}}(G, H) \tag{83}$$
$$(H \cdot \eta) \circ k \sqsubseteq (k \cdot UF) \circ (G \cdot \eta) \quad k \in Hom_{\mathbb{C} \Rightarrow \mathbb{E}}(G, H) \tag{84}$$

12.

$$((h \cdot U)^*)^{\otimes} \doteq h \cdot U \quad h \in Hom_{\mathbb{D} \Rightarrow \mathbb{E}}(G, H) \tag{85}$$
$$\left((k \cdot F)^{\otimes}\right)^* \doteq k \cdot F \quad k \in Hom_{\mathbb{C} \Rightarrow \mathbb{E}}(G, H) \tag{86}$$

or in reduced form with the same notations:

$$(H \cdot U \cdot \epsilon) \circ (h \cdot UFU) \circ (G \cdot \eta \cdot U) \doteq h \cdot U \tag{87}$$
$$(H \cdot \epsilon \cdot F) \circ (k \cdot FUF) \circ (G \cdot F \cdot \eta) \doteq k \cdot F \tag{88}$$

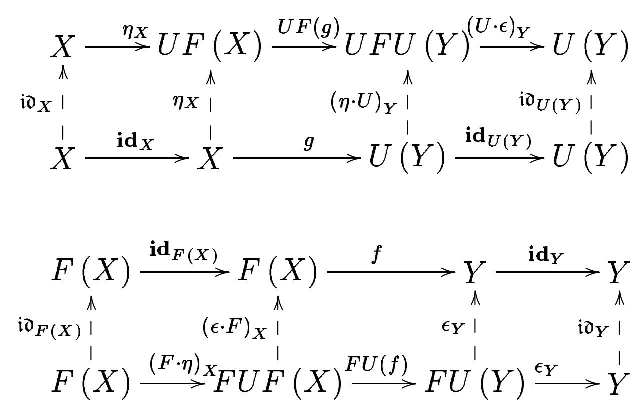
4.18. Example. [The double category Pfn of partial functions.] (see example 3.2, page 18 and example 4.1, page 22) By virtue of the latter lemma for the components of binary products $\langle \mathrm{rec}, \mathrm{succ} \rangle$

$$p_{\mathbb{R},\mathbb{R}} \circ \langle \mathrm{rec}, \mathrm{succ} \rangle \sqsubseteq \mathrm{rec} \tag{89}$$
$$q_{\mathbb{R},\mathbb{R}} \circ \langle \mathrm{rec}, \mathrm{succ} \rangle \sqsubseteq \mathrm{succ} \tag{90}$$

holds as expected.

Proof. It suffices to show the points 1 until 6, i. e. the points which relates to $[\_]$ and $\langle \_ \rangle$. The other points can be proven in analogous way.
Here the proofs are listed for the points of the lemma in the same order:

1.



2. $f \sqsubseteq \langle g \rangle$ implies under usage of the monotonicity of the map $[\_]$ with respect to $\sqsubseteq$ $[f] \sqsubseteq [\langle g \rangle]$. By virtue of the relation of the point 1 of our lemma which is proven immediately above and by virtue of the transitivity of the $\sqsubseteq$ relation the first part of the conjecture can be entailed. The second part of the conjecture can be entailed in dual way.

3. $\langle g\rangle \sqsubseteq \langle[\langle g\rangle]\rangle$ is an instance of the second relation of the point 1; $\langle[\langle g\rangle]\rangle \sqsubseteq \langle g\rangle$ follows from monotonicity of $\langle\_\rangle$ as instance of the first relation of the point 1.

4. Lemma 4.12 (page 25) implies:

$$\begin{aligned} U(h) &\doteq U(h)\circ(U\cdot\epsilon)_X\circ(\eta\cdot U)_X \\ &= \langle h\circ\epsilon_X\rangle \end{aligned} \tag{91}$$

   The other relation of the point follows in an analogous wise.

5. This is instance of point 1 of our lemma by application of the point 4 of our lemma

$$[\langle h\circ\epsilon_X\rangle] \sqsubseteq h\circ\epsilon_X \tag{92}$$

$$\eta_Y\circ k \sqsubseteq \langle[\eta_Y\circ k]\rangle \tag{93}$$

6. Point 3 and point 4 implies point 6.

■

4.19. Lemma. *Given two adjunctions $(F_1, U_1, \eta_1, \epsilon_1)$ between double categories $\mathbb{C}$ and $\mathbb{D}$ and $(F_2, U_2, \eta_2, \epsilon_2)$ between double categories $\mathbb{D}$ and $\mathbb{E}$. It holds:*

$$\eta_2\circ\epsilon_1 \doteq (\epsilon_1\cdot U_2\cdot F_2)\circ(F_1\cdot U_1\cdot\eta_2) \tag{94}$$

$$\doteq (U_2\cdot F_2\cdot\epsilon_1)\circ(\eta_2\cdot F_1\cdot U_1) \tag{95}$$

$$[U_1\cdot\eta_2]_1 \doteq \langle F_2\cdot\epsilon_1\rangle_2 \tag{96}$$

$$[\eta]_1 \doteq \eta_2\cdot F_1 \tag{97}$$

$$\langle\epsilon\rangle_2 \doteq U_2\cdot\epsilon_1 \tag{98}$$

The maps $[\_]$ or $\langle\_\rangle$, resp., of these both adjunctions are distinguished by respective subscripts in these equations. η and ϵ are determined as in lemma 4.14 (page 26).

Proof. The first of the relations you can get by application of the lemma 4.17 (page 27), points 5 and 11. The other are implications of it. For the latter two relation one needs lemma 4.12 (page 25). ■

4.20. Lemma. *Given an adjunction (F, U, η, ϵ) between the double categories $\mathbb{C}$ and $\mathbb{D}$. Then:*

$$\begin{array}{ccc} F(Y) & \xrightarrow{f} & A \\ {\scriptstyle F(\mathfrak{h})}\Uparrow & & \Uparrow{\scriptstyle \mathfrak{f}} \\ F(X) & \xrightarrow{F(h)} & F(Z) \end{array} \quad \textit{iff} \quad \begin{array}{ccc} Y & \xrightarrow{\langle f\rangle} & U(A) \\ {\scriptstyle \mathfrak{h}}\Uparrow & & \Uparrow{\scriptstyle \langle\mathfrak{f}\rangle} \\ X & \xrightarrow{h} & Z \end{array} \tag{99}$$

$$\begin{array}{ccc} U(Y) & \xrightarrow{U(f)} & U(Z) \\ {\scriptstyle \mathfrak{h}}\Uparrow & & \Uparrow{\scriptstyle U(\mathfrak{f})} \\ A & \xrightarrow{h} & U(X) \end{array} \quad \textit{iff} \quad \begin{array}{ccc} Y & \xrightarrow{f} & Z \\ {\scriptstyle [\mathfrak{h}]}\Uparrow & & \Uparrow{\scriptstyle \mathfrak{f}} \\ F(A) & \xrightarrow{[h]} & X \end{array} \tag{100}$$

$$
\begin{array}{ccc}
B & \xrightarrow{\langle f\rangle} & U(Y) \\
{\scriptstyle\mathfrak{h}}\uparrow & & \uparrow{\scriptstyle U(\mathfrak{k})} \\
A & \xrightarrow{h} & U(X)
\end{array}
\quad \text{iff} \quad
\begin{array}{ccc}
F(B) & \xrightarrow{f} & Y \\
{\scriptstyle F(\mathfrak{h})}\uparrow & & \uparrow{\scriptstyle \mathfrak{k}} \\
F(A) & \xrightarrow{[h]} & X
\end{array}
\tag{101}
$$

$$
\begin{array}{ccc}
U(X) & \xrightarrow{U(f)} & U(Y) \\
{\scriptstyle\mathfrak{h}}\uparrow & & \uparrow{\scriptstyle \langle\mathfrak{f}\rangle} \\
A & \xrightarrow{h} & B
\end{array}
\quad \text{iff} \quad
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
{\scriptstyle [\mathfrak{h}]}\uparrow & & \uparrow{\scriptstyle \mathfrak{f}} \\
F(A) & \xrightarrow{F(h)} & F(B)
\end{array}
\tag{102}
$$

Proof. We confine ourselves. The second statement is the dual of the first. The fourth is the dual of the third.

- *the first statement.*

$$
\begin{array}{ccccc}
Y & \xrightarrow{\eta_Y} & UF(Y) & \xrightarrow{U(f)} & U(A) \\
{\scriptstyle\mathfrak{h}}\uparrow & & \uparrow{\scriptstyle UF(\mathfrak{h})} & & \uparrow{\scriptstyle U(\mathfrak{f})} \\
X & \xrightarrow{\eta_X} & UF(X) & \xrightarrow{UF(h)} & UF(Z) \\
{\scriptstyle\mathfrak{id}_X}\uparrow & & \uparrow{\scriptstyle \eta_X} & & \uparrow{\scriptstyle \eta_Z} \\
X & \xrightarrow{\mathbf{id}_X} & X & \xrightarrow{h} & Z
\end{array}
$$

$$
\begin{array}{ccccccc}
F(Y) & \xrightarrow{\mathbf{id}_{F(Y)}} & F(Y) & \xrightarrow{f} & A & \xrightarrow{\mathbf{id}_A} & A \\
{\scriptstyle\mathfrak{id}_{F(Y)}}\uparrow & & \uparrow{\scriptstyle (\epsilon\cdot F)_Y} & & \uparrow{\scriptstyle \epsilon_A} & & \uparrow{\scriptstyle \mathfrak{id}_A} \\
F(Y) & \xrightarrow{(F\cdot\eta)_Y} & FUF(Y) & \xrightarrow{FU(f)} & FU(A) & \xrightarrow{\epsilon_A} & A \\
\uparrow & & & & \uparrow{\scriptstyle FU(\mathfrak{f})} & & \uparrow{\scriptstyle \mathfrak{f}} \\
{\scriptstyle F(\mathfrak{h})}\mid & & & & FUF(Z) & \xrightarrow{(\epsilon\cdot F)_Z} & F(Z) \\
\mid & & & & \uparrow{\scriptstyle (F\cdot\eta)_Z} & & \uparrow{\scriptstyle \mathfrak{id}_{F(Z)}} \\
F(X) & & \xrightarrow{F(h)} & & F(Z) & \xrightarrow{\mathbf{id}_{F(Z)}} & F(Z)
\end{array}
$$

- *the third statement.*
  By virtue of lemma 4.17, point 1.

$$
\begin{array}{ccccc}
F(B) & & \xrightarrow{f} & & Y \\
{\scriptstyle\mathfrak{id}_{F(B)}}\uparrow & & & & \uparrow{\scriptstyle \mathfrak{id}_Y} \\
F(B) & \xrightarrow{F(\langle f\rangle)} & FU(Y) & \xrightarrow{\epsilon_Y} & Y \\
{\scriptstyle F(\mathfrak{h})}\uparrow & & \uparrow{\scriptstyle FU(\mathfrak{f})} & & \uparrow{\scriptstyle \mathfrak{f}} \\
F(A) & \xrightarrow{F(h)} & FU(X) & \xrightarrow{\epsilon_X} & X
\end{array}
\qquad
\begin{array}{ccccc}
B & \xrightarrow{\eta_B} & UF(B) & \xrightarrow{U(f)} & U(Y) \\
{\scriptstyle\mathfrak{h}}\uparrow & & \uparrow{\scriptstyle UF(\mathfrak{h})} & & \uparrow{\scriptstyle U(\mathfrak{f})} \\
A & \xrightarrow{\eta_A} & UF(A) & \xrightarrow{U([h])} & U(X) \\
{\scriptstyle\mathfrak{id}_A}\uparrow & & & & \uparrow{\scriptstyle \mathfrak{id}_{U(X)}} \\
A & & \xrightarrow{h} & & U(X)
\end{array}
$$

■

4.21. Lemma. *Given an adjunction* (F, U, η, ϵ) *between double categories* $\mathbb{C}$ *and* $\mathbb{D}$. η (ϵ) *is a natural transformation between the partial categories (vertical and horizontal, resp.)* [4] *factorized by the equimorphism relation iff for each* $h \in Hom_{\mathbb{D}}(F(X), A)$ $(k \in Hom_{\mathbb{C}}(X, U(A)))$ $\langle[h]\rangle \doteq h$ $([\langle k\rangle] \doteq k)$ *holds.*

Proof. To prove we confine ourselves to one of the dual statements.

- *Only if.*
  Using lemma 4.12 (page 25):

$$\begin{aligned} \langle[h]\rangle &= (U \cdot \epsilon)_A \circ UF(h) \circ \eta_X && (103)\\ &\doteq (U \cdot \epsilon)_A \circ (\eta \cdot U)_A \circ h && (104)\\ &\doteq h && (105) \end{aligned}$$

- *If.*

$$\begin{aligned} UF(x) \circ \eta_X &= \langle F(x)\rangle && (106)\\ &\doteq \langle[\eta_Y \circ x]\rangle && (107)\\ &\doteq \eta_Y \circ x && (108) \end{aligned}$$

■

4.22. Lemma. *Given an adjunction* (F, U, η, ϵ) *between double categories* $\mathbb{C}$ *and* $\mathbb{D}$. *Let hold for the paramorphism* η (ϵ): $\eta \circ \eta_* \doteq id_{UF}$ $(\epsilon_* \circ \epsilon \doteq id_{FU})$ *and* $\eta_* \circ \eta \doteq id_{Id_{\mathbb{C}}}$ $(\epsilon_* \circ \epsilon \doteq id_{Id_{\mathbb{D}}})$ *with a inversion paramorphism* η_* (ϵ_*). *It is valid:*

- $\eta \in |Id_{\mathbb{C}} \Rightarrow UF|$, $(\epsilon \in |FU \Rightarrow Id_{\mathbb{D}}|)$ *and* $\eta_* \in |UF \Rightarrow Id_{\mathbb{C}}|$ $(\epsilon_* \in |Id_{\mathbb{D}} \Rightarrow FU|)$ *are isomorphisms.*
- $U \cdot \epsilon = \eta_* \cdot U$ *and* $\epsilon \cdot F = F \cdot \eta_*$ $(F \cdot \eta = \epsilon_* \cdot F$ *and* $\eta \cdot U = U \cdot \epsilon_*)$.
- *There are the following squares:*

$$\begin{array}{ccc} FUF & \xrightarrow{\mathbf{id}_{FUF}} & FUF \\ {\scriptstyle \mathrm{id}_{FUF}}\uparrow & & \uparrow{\scriptstyle F\cdot\eta} \\ FUF & \xrightarrow{\epsilon\cdot F} & F \end{array} \qquad \begin{array}{ccc} UFU & \xrightarrow{\mathbf{id}_{UFU}} & UFU \\ {\scriptstyle \mathrm{id}_{UFU}}\uparrow & & \uparrow{\scriptstyle \eta\cdot U} \\ UFU & \xrightarrow{U\cdot\epsilon} & U \end{array}$$

$$\begin{array}{ccc} F & \xrightarrow{F\cdot\eta} & FUF \\ {\scriptstyle \epsilon\cdot F}\uparrow & & \uparrow{\scriptstyle \mathrm{id}_{FUF}} \\ FUF & \xrightarrow{\mathbf{id}_{FUF}} & FUF \end{array} \qquad \begin{array}{ccc} U & \xrightarrow{\eta\cdot U} & UFU \\ {\scriptstyle U\cdot\epsilon}\uparrow & & \uparrow{\scriptstyle \mathrm{id}_{UFU}} \\ UFU & \xrightarrow{\mathbf{id}_{UFU}} & UFU \end{array}$$

[4] i. e. $\eta_Y \circ x \doteq UF(x) \circ \eta_X$ holds for each $x \in \mathrm{Hom}_{\mathbb{C}}(X, Y)$ $(a \circ \epsilon_A \doteq \epsilon_B \circ FU(a)$ for each $a \in \mathrm{Hom}_{\mathbb{D}}(A, B))$.

Proof. The proof is done only for one of the dual cases.

- $U \cdot \epsilon = \eta_* \cdot U$, $\epsilon \cdot F = F \cdot \eta_*$.
  Using lemma 4.12 (page 25) you can derive:

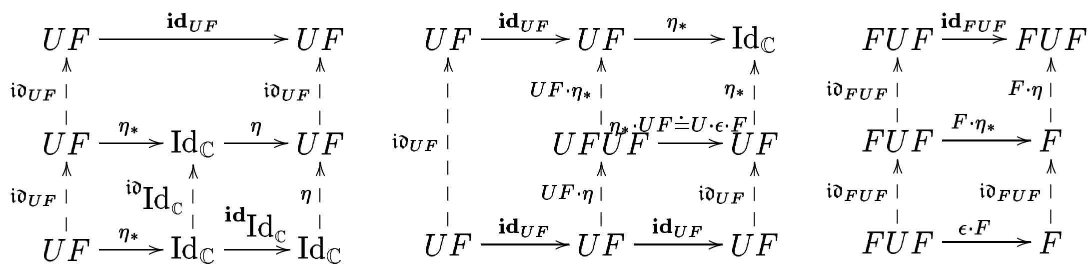
$$(U \cdot \epsilon) \circ (\eta \cdot U) \doteq \mathrm{id}_U \tag{109}$$

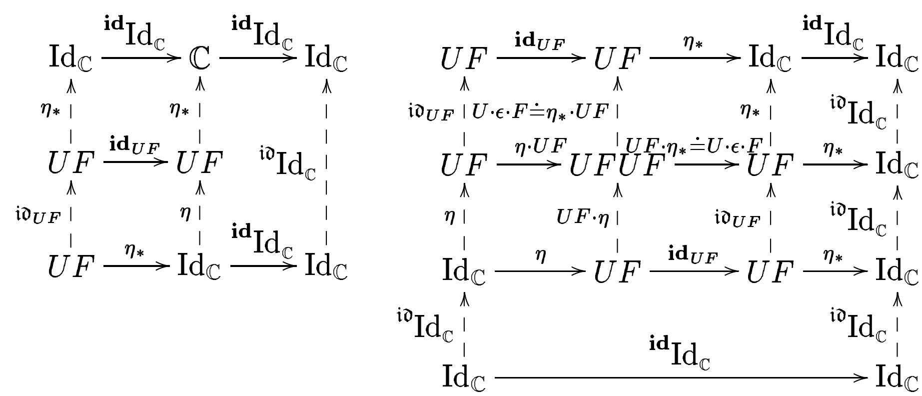
$$(U \cdot \epsilon) \circ (\eta \cdot U) \circ (\eta_* \cdot U) \doteq \mathrm{id}_U \circ (\eta_* \cdot U) \tag{110}$$

$$U \cdot \epsilon \doteq \eta_* \cdot U \tag{111}$$

  Similarly for the other relation, too.

- *The squares.*





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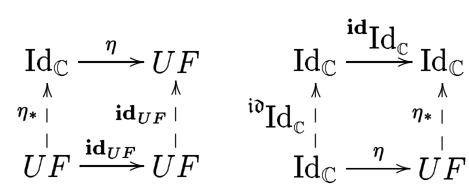
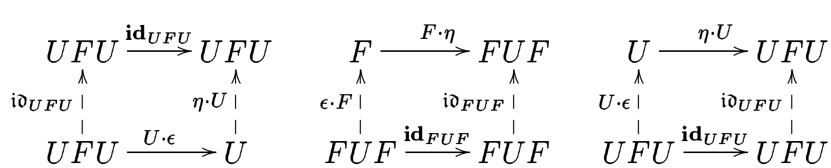




can be proven.

■

4.23. Lemma.

- *Given two adjunctions* $(F_1, U, \eta_1, \epsilon_1)$ $((F, U_1, \eta_1, \epsilon_1))$ *and* $(F_2, U, \eta_2, \epsilon_2)$ $((F, U_2, \eta_2, \epsilon_2))$ *between double categories* $\mathbb{C}$ *and* $\mathbb{D}$*. Then:*

$$F_1 \quad \cong \quad F_2 \tag{112}$$

or

$$U_1 \quad \cong \quad U_2 \tag{113}$$

resp., with isomorphism

$$\begin{array}{ccc} \mathbb{D} & & \mathbb{D} \\ {\scriptstyle F_1}\Big\Uparrow & \xleftrightarrow[\kappa_*]{\kappa} & \Big\Uparrow{\scriptstyle F_2} \\ \mathbb{C} & & \mathbb{C} \end{array} \quad \text{or} \quad \begin{array}{ccc} \mathbb{C} & & \mathbb{C} \\ {\scriptstyle U_1}\Big\Uparrow & \xleftrightarrow[\kappa_*]{\kappa} & \Big\Uparrow{\scriptstyle U_2} \\ \mathbb{D} & & \mathbb{D} \end{array} \quad \text{resp.,}$$

in the first case

$$\begin{aligned} \kappa &:= (\epsilon_1 \cdot F_2) \circ (F_1 \cdot \eta_2) && (114) \\ \kappa_* &:= (\epsilon_2 \cdot F_1) \circ (F_2 \cdot \eta_1) && (115) \end{aligned}$$

and in the other case

$$\begin{aligned} \kappa &:= (U_2 \cdot \epsilon_1) \circ (\eta_2 \cdot U_1) && (116) \\ \kappa_* &:= (U_1 \cdot \epsilon_2) \circ (\eta_1 \cdot U_2) && (117) \end{aligned}$$

- *The following relations hold:*

$$\begin{aligned} \eta_2 &\doteq \kappa_\eta \circ \eta_1 && (118) \\ \eta_1 &\doteq \kappa_{*,\eta} \circ \eta_2 && (119) \\ \epsilon_1 \circ \kappa_\epsilon &\doteq \epsilon_2 && (120) \\ \epsilon_2 \circ \kappa_{*,\epsilon} &\doteq \epsilon_1 && (121) \end{aligned}$$

where in the first case it holds:

$$\begin{aligned} \kappa_\eta &= U \cdot \kappa && (122) \\ \kappa_{*,\eta} &= U \cdot \kappa_* && (123) \\ \kappa_\epsilon &= \kappa \cdot U && (124) \\ \kappa_\epsilon &= \kappa_* \cdot U && (125) \end{aligned}$$

and in the second case it holds:

$$\kappa_\eta := \kappa \cdot F \tag{126}$$
$$\kappa_{*,\eta} := \kappa_* \cdot F \tag{127}$$
$$\kappa_\epsilon := F \cdot \kappa \tag{128}$$
$$\kappa_\epsilon := F \cdot \kappa_* \tag{129}$$

- *If* $F := F_1 = F_2$, $U := U_1 = U_2$ *holds, so depending on the case*

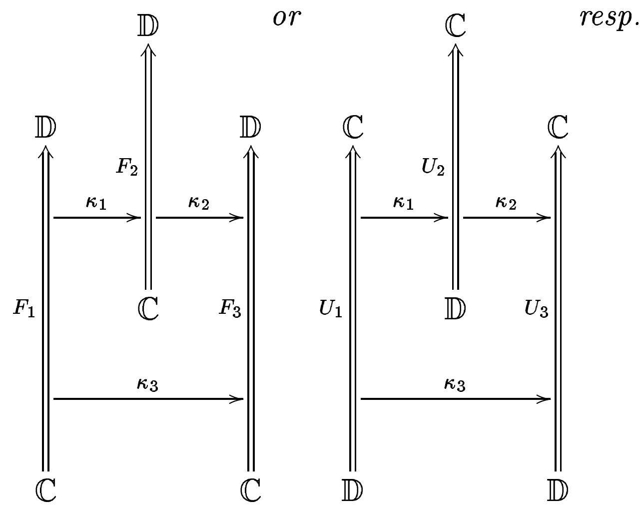
$$\kappa \doteq id_F \tag{130}$$

or

$$\kappa \doteq id_U \tag{131}$$

resp., holds.

- *Given three adjunctions* $(F_1, U, \eta_1, \epsilon_1)$ $((F, U_1, \eta_1, \epsilon_1))$, $(F_2, U, \eta_2, \epsilon_2)$ $((F, U_2, \eta_2, \epsilon_2))$ *and* $(F_3, U, \eta_3, \epsilon_3)$ $((F, U_3, \eta_3, \epsilon_3))$ *between double categories* $\mathbb{C}$ *and* $\mathbb{D}$. $\kappa_1 \in Hom_{\mathbb{C}\Rightarrow\mathbb{D}}(F_1, F_2)$ $(\kappa_1 \in Hom_{\mathbb{D}\Rightarrow\mathbb{D}}(U_1, U_2))$, $\kappa_2 \in Hom_{\mathbb{C}\Rightarrow\mathbb{D}}(F_2, F_3)$ $(\kappa_2 \in Hom_{\mathbb{D}\Rightarrow\mathbb{D}}(U_2, U_3))$, $\kappa_3 \in Hom_{\mathbb{C}\Rightarrow\mathbb{D}}(F_1, F_3)$ $(\kappa_3 \in Hom_{\mathbb{D}\Rightarrow\mathbb{D}}(U_1, U_3))$ *are the isomorphisms which are constructed in the first point of this lemma:*



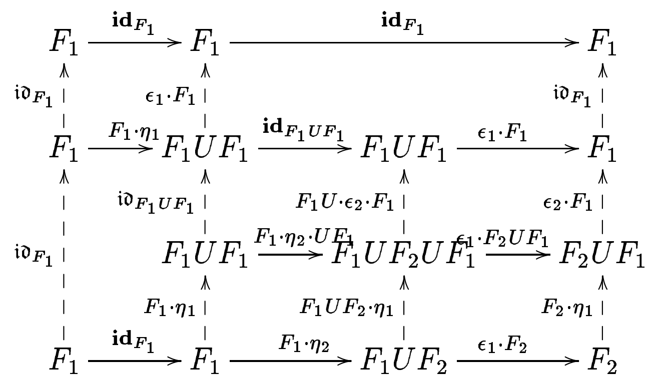
It holds:

$$\kappa_3 \doteq \kappa_2 \circ \kappa_1 \tag{132}$$

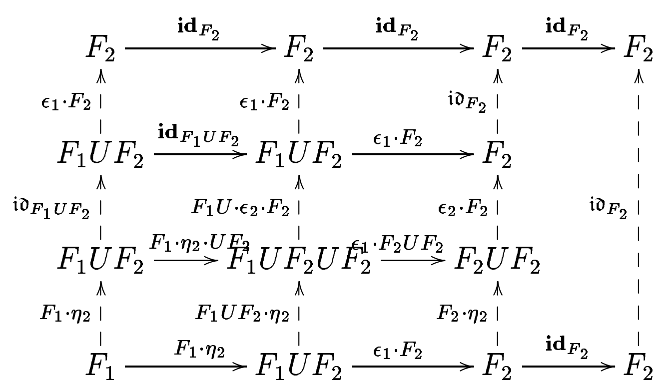
- *Given an adjunction* (F, U, η, ϵ) *and isomorphisms* $\kappa \in Hom_{\mathbb{C}\Rightarrow\mathbb{D}}(F, F')$ / $\kappa_* \in Hom_{\mathbb{C}\Rightarrow\mathbb{D}}(F', F)$ $(\kappa \in Hom_{\mathbb{D}\Rightarrow\mathbb{C}}(U, U')$ / $\kappa_* \in Hom_{\mathbb{D}\Rightarrow\mathbb{C}}(U', U)$ $)$ *for functors* F' (U'). *Then:* $(F', U, (U \cdot \kappa) \circ \eta, \epsilon \circ (\kappa_* \cdot U))$ $((F, U', (\kappa \cdot F) \circ \eta, \epsilon \circ (F \cdot \kappa_*))$ $)$ *are adjunctions, too, and* κ / κ_* *is the isomorphism determined as in den last points.*

Proof. The third statement of the lemma is the content of the lemma 4.12 (page 25). The last statement of the lemma are yielded just by fitting in the respective definitions and application of the lemma 4.17 (page 27).

- *The first statement.*
  The existence of squares of the definition 4.7 (page 23) is the only one has to prove. We only prove the existence of two of these squares. The other are dual to it. To prove the existence of the other squares we confine ourselves to two cases, the other cases entails in the dual situation.



  A second square can be proven by using lemma 4.12 (page 25):



- *The second statement.*
  By virtue of lemma 4.17 (page 27) the proof of the second statement is done. We restrict our attention to one of the many dual cases:

$$\eta_2 \sqsubseteq \langle [\eta_2]_1 \rangle_1 \quad (133)$$

$$= \kappa_\eta \circ \eta_1 \quad (134)$$

  In similar way it follows:

$$\eta_1 \sqsubseteq \kappa_{*,\eta} \circ \eta_2 \quad (135)$$

and further it follows:

$$\kappa_\eta \circ \eta_1 \sqsubseteq \kappa_\eta \circ \kappa_{*,\eta} \circ \eta_2 \tag{136}$$
$$\doteq \eta_2 \tag{137}$$

The dual statements can be proven in the similar way.

- *The fourth statement.*
One case of the fourth statement is completely dual to the other case of it. That is why it suffices to prove one of the cases:
Lemma 4.17 (page 27) implies:

$$\begin{aligned} \eta_2 &\sqsubseteq (U \cdot \kappa_1) \circ \eta_1 \\ &= (U \cdot \epsilon_1 \cdot F_2) \circ (UF_1 \cdot \eta_2) \circ \eta_1 \end{aligned} \tag{138}$$
$$\begin{aligned} \eta_3 &\sqsubseteq (U \cdot \kappa_2) \circ \eta_2 \\ &= (U \cdot \epsilon_3 \cdot F_3) \circ (UF_2 \cdot \eta_3) \circ \eta_2 \end{aligned} \tag{139}$$

By fitting in to each other you can imply the following relations by transitivity of the relation $\sqsubseteq$:

$$\begin{aligned} \eta_3 &\sqsubseteq (U \cdot \kappa_2) \circ (U \cdot \kappa_1) \circ \eta_1 \\ &= \langle \kappa_2 \circ \kappa_1 \rangle_1 \end{aligned} \tag{140}$$

Using the point 2 of the lemma 4.17 (page 27) a part of the conjecture of the lemma follows:

$$\begin{aligned} (\epsilon_1 \cdot F_3) \circ (F_1 \cdot \eta_3) &\doteq \\ \kappa_3 &\sqsubseteq \kappa_2 \circ \kappa_1 \end{aligned} \tag{141}$$

The other part of the assertion of this lemma you can get when you redo the above proof in the dual situation, i. e.

$$\epsilon_2 \circ (\kappa_1 \cdot U) \sqsubseteq \epsilon_1 \tag{142}$$
$$\epsilon_3 \circ (\kappa_2 \cdot U) \sqsubseteq \epsilon_2 \tag{143}$$

and

$$(\kappa_2 \circ \kappa_1)^{\otimes 3} \doteq \tag{144}$$
$$\epsilon_3 \circ (\kappa_2 \cdot U) \circ (\kappa_1 \cdot U) \sqsubseteq \epsilon_1 \tag{145}$$

and using point 8 of lemma 4.17 (page 27):

$$\kappa_2 \circ \kappa_1 \sqsubseteq (\epsilon_1 \cdot F_3) \circ (F_1 \cdot \eta_3) \tag{146}$$
$$\doteq \kappa_3 \tag{147}$$

■

4.24. Corollary. *Given a double category with finite (co)-products.* $\prod^{n}$ *denotes the n-(co)-product-functor and we define* $A_1 \times \ldots \times A_n := \prod_{i:1}^{n} A_i$ *The following isomorphisms hold:*

commutativity.

$$\prod_{i:1}^{n} A_i \;\cong\; \prod_{i:1}^{n} A_{p(i)} \tag{148}$$

with p an arbitrary permutation on $\{1, \ldots, n\}$*;*

associativity.

$$\prod_{i:1}^{m} \left(\prod_{j:1}^{m_i} A_{ij} \right) \;\cong\; A_{11} \times \ldots \times A_{1m_1} \times \ldots \ldots \times A_{m1} \times \ldots \times A_{mm_m} \tag{149}$$

Proof. The proof you can get by virtue of lemma 4.23 (page 34). We construct the other adjunction, which we need for application of this lemma, by virtue of lemma 4.14 (page 26) using an adjunction $\left(K, K_*, \mathrm{id}_{\mathrm{Id}_{\mathbb{E}}}, \mathrm{id}_{\mathrm{Id}_{\mathbb{C}}}\right)$, i. e. an category equivalence, between a double category $\mathbb{E}$ and a double category $\mathbb{C}$. For the commutativity law we take $K := \left\{(A_1, \ldots, A_n) \mapsto \left(A_{p(1)}, \ldots, A_{p(n)}\right)\right\}$. One readily verifies, $K_* \circ (\_)_n = (\_)_n$. To prove the associativity law we take: $K := \{((A_{11}, \ldots, A_{1,m_1}), \ldots, (A_{m1}, \ldots, A_{m,m_m})) \mapsto (A_{11}, \ldots, A_{mm_m})\}$. ∎

5. Conclusion

5.1. Related work.

Ştefănescus *flows* ([Ş94]), Scedrovs / Freyds *upas* ([FS90]) and [BH94]. Another axiomatizations of flow graphs like those made by Ştefănescu with his *flows* and made by Scedrov/ Freyd with their *upas* can be derived by the data flow logic under the same assumptions like those in subsection 2.29 (page 14) or a slight extension of them. A characteristic feature of the approach from this paper is, in opposition to both former axiomatizations, that an abstract operational semantics is defined from which one can derive the equation relation which is the base of the axiomatizations both of Ştefănescu and also of Scedrov/ Freyd.
Interestingly, the proof calculus in [BH94] is very closely related to the double-categorical approach which is the subject of this paper.

Gadducci/ Montanaris 'tile model' ([GM96]). The approach to formalize operational semantics of flow graphs which is spoken about in this paper turns out to be closely related to work done by Gadducci/ Montanari who use double categories to formalize special kinds of rewrite theories which are used to describe operational semantics of models of concurrency.

5.2. Summary. Now, we saw that interesting categorical constructions can be defined without any equation. As shown by the examples 3.2 (page 18), 3.3 (page 18) and 3.4 (page 19) these constructions are in many double categories used in computer science whose definition in conventionally categorical manner is not as easy. For these constructions we could prove some properties which one knows from adjunctions like the composition of adjunction (see lemma 4.14, page 26), determinacy up-to isomorphism (see lemma 4.23, page 34), commutativity and associativity of products (see corollary 4.24 page 38).

5.3. Acknowledgments. I am grateful to Stefan Conrad for the fruitful discussion on this paper.

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